**1.** (a) At a recent cat-show there were 20 entries, of which 8 were longhaired and the rest were shorthaired. Prizes are awarded for first, second and third of each type.

(i) How many possible ordered choices of prizewinners are there?

The main competition is for an overall first, second and third prize. The rules specify that the three prizewinners can not all be longhaired or all shorthaired.

(ii) How many ordered choices of prizewinners are possible under these rules?

As a preliminary help in judging, the cats were divided into four groups of five (irrespective of hair-length) and a first and second in each group were chosen.

(iii) In how many ways can they be divided into groups?

(iv) How many eventual choices of group prizewinners are there, where the winning cats are simply given rosettes saying 'group winner' or 'group second'?

(v) Recalculate these numbers, given that there were five Siamese cats in the show, and the judges decided to treat them as one of the groups.

(b) Show that in every set $S$ of six numbers drawn from $\{1, 2, \ldots, 9\}$ there will be two subsets of $S$ with three numbers in each which have the same sum.

[20 marks]

**2.** (i) Write $s_k$ for the number of solutions of the equation $a+b+c=k$ in non-negative integers $a, b, c$. Give a formula for $s_k$ and hence find the generating function $S(x)$ defined by

$$S(x) = \sum_{k=0}^{\infty} s_k x^k.$$

(ii) Find the number of terms in the expansion of $(u+v+w)^7$. Give also the coefficient of $u^2v^3w^2$ in this expansion.

(iii) Write $d_k$ for the number of solutions to the equation $a+b+c=k$ with $a \geq A, b \geq 0, c \geq 0$.

Show that $\sum_{k=0}^{\infty} d_k x^k = x^A S(x)$. Hence show that the generating function for $e_k$, the number of solutions to $a+b+c=k$ with $0 \leq a \leq A-1, b \geq 0, c \geq 0$, is given by

$$\sum_{k=0}^{\infty} e_k x^k = (1-x^A)S(x).$$

(iv) Show how this generating function is modified when the condition $b \geq 0$ is replaced by $b \geq B$.

Deduce that the number of integer solutions $r_k$ to $a+b+c=k$ subject to $0 \leq a \leq A-1, 0 \leq b \leq B-1, 0 \leq c \leq C-1$ is given by

$$\sum_{k=0}^{\infty} r_k x^k = (1-x^A)(1-x^B)(1-x^C)S(x).$$

(v) Hence or otherwise find the number of integer solutions to the equation $a+b+c=22$ with $3 \leq a \leq 10,\ 4 \leq b \leq 11$ and $5 \leq c \leq 12$.

[20 marks]

**3.** State Hall's Assignment Theorem.

(a) For each of the pruned chessboards shown below, where ■ denotes a square that has been deleted, construct a family of sets which has a system of distinct representatives if and only if there is a perfect cover by $2 \times 1$ tiles. Decide whether the chessboard has a perfect cover or not.

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|---|---|
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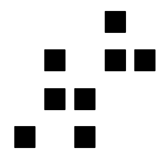
(b) A headteacher has to allocate eight duties among seven staff. The staff, $A, B, \ldots, G$, are able to do certain of the duties, as listed. Duty 1 can be done by A,B,C,E; duty 2 by A,B,D; duty 3 by A,D; duty 4 by B,C,D,E,G; duty 5 by B,D,F; duty 6 by A,B,F; duty 7 by E,F,G; and duty 8 by D,F. Show how to allocate the duties so that only one member of staff has to do two.

To help out, the head volunteers to do duty 7. Does this allow a reallocation of duties with everybody doing just one? [20 marks]

**4.** (i) Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula.

Calculate the rook polynomial of the $4 \times 5$ board shown.



(ii) Five friends, Arthur, Betty, Charles, Diane and Edward, have gone out for a late lunch. All that is left on the menu is one dish each of beef, chicken, duck, eggs and fish.

Betty doesn't like beef, Charles doesn't like chicken and Edward doesn't like eggs. Neither Betty nor Charles wants duck, while Diane only likes duck and beef.

In how many ways can they distribute the dishes so that each person has a dish that they will eat?

In how many more ways can the lunch be arranged if Charles decides that he is prepared to eat the duck? [20 marks]

**5.** (a) Solve the following recurrence relations. In each case you should find an expression for $a_n$ and also an expression for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i) $a_{n+2} = 2a_{n+1} + 3a_n, n \geq 0,\ a_0 = a_1 = 1.$

(ii) $a_{n+1} = 2a_n + 3^n, n \geq 0,\ a_0 = 1.$

(iii) $a_{n+2} = 2a_{n+1} - a_n + 1, n \geq 0,\ a_0 = a_1 = 1.$

(b) Let $B_n$ be the $n \times n$ matrix with entries 2 on the diagonal, $-1$ immediately above the diagonal and 3 immediately below the diagonal, all other entries being 0. Write $b_0 = 1$ and $b_n = \det(B_n), n \geq 1$, so that, for example,

$$b_5 = \det \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 3 & 2 & -1 & 0 & 0 \\ 0 & 3 & 2 & -1 & 0 \\ 0 & 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & 3 & 2 \end{pmatrix}.$$

Find a recurrence relation for $b_{n+2}$ in terms of $b_{n+1}$ and $b_n$. Hence find an expression for $B(x) = \sum_{n=0}^{\infty} b_n x^n$, and an expression for $b_n$. [20 marks]

**6.** State the inclusion-exclusion formula. Hence obtain a formula for the number $D_n$ of derangements of $n$ objects. Show that

$$\frac{e^{-x}}{1-x} = 1 + \sum_{n=1}^{\infty} \frac{D_n}{n!} x^n.$$

Calculate $D_n$ for $1 \leq n \leq 6$. Find a formula for the number of permutations of $n$ objects which leave just $r$ of the objects unmoved. Calculate these numbers for $n = 6$ and $0 \leq r \leq 6$, and check that they add up to 6!.

List the four different cycle types of the derangements of six objects. Find the number of permutations of each of these cycle types and check that they add up to $D_6$. [20 marks]

**7.** Obtain a formula for the generating function $S(x) = \sum_{n=1}^{\infty} s_n t^n$, where $s_n$ is the number of solutions of $n = 2a + 3b + 6c$ in non-negative integers.

Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer $n$. Find also the function $P_o(t)$ which enumerates the number of partitions of $n$ into odd parts, and the function $P_d(t)$ to enumerate the partitions of $n$ with distinct parts. For any positive integer $n$, show that the number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.

Define the term *Ferrers graph.* Use Ferrers graphs to establish a bijection between self-conjugate partitions of $n$ and partitions with distinct odd parts.

Write down the generating function which enumerates such partitions. Hence determine the number of self-conjugate partitions of 13. Exhibit the corresponding Ferrers graphs.

**8.** Define the term *symmetric function.* For any positive integer $n$, define the *elementary symmetric function* $\sigma_n$ and the *power sum symmetric function* $\pi_n$. State and prove the Newton Identities.

Express $\pi_3$ in terms of the elementary symmetric functions and $\sigma_3$ in terms of the power sum functions.

Obtain, in terms of the elementary symmetric functions of $\alpha, \beta$ and $\gamma$,

(i) the elementary symmetric functions of $1/\alpha, 1/\beta, 1/\gamma$, and

(ii) the equation with roots $\alpha^2,\ \beta^2,\ \gamma^2$.

Write $\delta_r$ for the determinant

$$\det \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^r & \beta^r & \gamma^r \end{vmatrix}$$

Show that, for each $r \geq 3$, $\phi_r = \delta_r/\delta_2$ is a symmetric function of $\alpha, \beta$ and $\gamma$. Express $\phi_4$ in terms of their elementary symmetric functions.

[20 marks]