1. Define a *group*. Let X and Y be the 2×2 matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where i denotes the complex square root of -1. Find the matrix inverse of each of X and Y. Find also the smallest integers r and s such that $X^r = Y^s = I$. Determine the group table for $G = \langle X, Y \rangle$ under matrix multiplication. Is G abelian? Find the non-identity 2×2 invertible matrix Z which satisfies the three conditions that $XZ = ZX$, $ZY = YZ$ and $Z^2 = I$. Does Z commute with every element of G?

2. State Lagrange's Theorem and use it to show that a group G with p elements (where p is a prime) is cyclic. Deduce that if H is a subgroup of G with p elements and K is a subgroup of G with q elements, where p and q are distinct prime numbers, then $H \cap K = \{1\}$.

Now let G be the dihedral group of symmetries of a hexagon (a regular 6-sided polygon). Thus

$$G = \{1, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$$

where x corresponds to rotation through 60 degrees and y corresponds to a reflection. You may assume that $yx = x^{-1}y$. Prove, by induction on i that $yx^i = x^{-i}y$. Use this fact to find the order of each element of G.

List the divisors of 12, and find a number d in your list such that G does not have an element of order d. Does G have a subgroup with d elements?

Assuming that G has a proper non-abelian subgroup H, how many elements are there in H?

3. Show that if G is any group and H is a subgroup of G, then two (left) cosets of H in G are either equal or have no elements in common.

Let G be the set of 12 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}; \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}; \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

You may assume (without proof) that these matrices form a group. Show that

$$H = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\}$$

is a subgroup of G. Calculate the complete list of distinct left cosets of H in G and also the complete list of distinct right cosets of H in G. Deduce that H is a normal subgroup of G and decide whether or not the quotient group G/H is cyclic.

Decide, giving your reasons, whether or not the set

$$K = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\}$$

is a normal subgroup of G.

4. Let ϑ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that ϑ is a homomorphism. Show that if ϑ is a homomorphism then $\vartheta(1_G) = 1_H$. Show also that if g and h are elements of G with h being the inverse of g (with respect to the operation $\circ$), then $\vartheta(h)$ is the inverse of $\vartheta(g)$ (with respect to the operation $*$). Define the kernel and the image of ϑ, and prove that the kernel of ϑ is a normal subgroup of G. State the homomorphism theorem.

Let G be the set of 4×4 matrices of the form

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where a, b, c are elements of $\mathbf{Z}$, the set of integers. You may assume, without proof, that G is a group under matrix multiplication. Prove that the map $\vartheta : G \to \mathbf{Z}$ defined by $\vartheta(A) = a$ is a group homomorphism (using addition for the group operation in $\mathbf{Z}$). Show also that the map $\phi : G \to \mathbf{Z}$ defined by $\phi(A) = b$ is not a group homomorphism. Calculate the kernel of ϑ and deduce that G has an abelian normal subgroup with cyclic factor group.

5. Give rules which enable the sign of a permutation to be determined. Show that the set of even permutations on n symbols forms a normal subgroup, $A(n)$, of the symmetric group $S(n)$. Is the set of odd permutations a subgroup?

Suppose that a permutation π is written as a product of disjoint cycles, express the order of π in terms of the lengths of these disjoint cycles.

Show that a permutation of odd order in $S(n)$ is even.

Determine the orders of the elements of each of the symmetric groups $S(4), S(5)$ and $S(6)$.

Find the smallest n such that $S(n)$ has an element of order 6 and find the smallest n such that $S(n)$ has an even element of order 10.

6. Let G be a group. Define the terms *G-set*, *orbit* and *stabilizer*. State the orbit-stabilizer theorem.

Show that G is itself a G-set under conjugation (so $g \circ x = gxg^{-1}$) and give explicit descriptions of the orbit of an element g of G and also of the stabilizer of g in this case.

Let G be the alternating group $A(4)$. Determine the complete list of conjugacy classes for G.

7. State the Sylow theorems and show that a group G has a unique Sylow p-subgroup if and only if the Sylow p-subgroups of G are normal.

Prove the following:

1. A group with 15 elements is cyclic;
2. A group with 12 elements either has a normal Sylow 2-subgroup or a normal Sylow 3-subgroup;
3. Give examples of three groups with 8 elements no two of which are isomorphic (giving brief reasons why no two of your chosen groups are isomorphic).

8. State the Jordan-Hölder Theorem explaining the terms you use. Find composition series for each of the following:

(1) a cyclic group of order 4;

(2) a non-cyclic group of order 4;

(3) a cyclic group of order 10;

(4) the symmetric group $S(4)$;

(5) the dihedral group $D(6)$.