Math343 Group Theory May 2005

Throughout this paper, $D(n)$ denotes the dihedral group of symmetries of a regular n-sided polygon. Thus $D(n)$ has $2n$ distinct elements and may be generated by an element a of order n together with an element b of order 2, with the relation $ab = ba^{-1}$. Then the distinct elements of $D(n)$ are the n powers of a together with n elements of the form b times a power of a. You may also assume throughout that for $0 \leq i \leq n-1$, we have $b^{-1}a^i b = a^{-i}$.

Also $S(n)$ denotes the group of $n!$ permutations on n symbols and $A(n)$ is its subgroup consisting of even permutations.

1. Define a *group.* Let θ be the permutation

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Find another permutation ϕ on $\{1, 2, 3, 4\}$, such that $\phi(1) = 1$ and $\theta\phi = \phi\theta^{-1}$. Now let G be the group generated by θ and ϕ. Write down 8 permutations in G, and show that these 8 permutations actually form the group G. Show that there is a non-trivial element z in G such that $zg = gz$ for every element g of G.

2. Let u and v be elements of a group G. Prove that the equation $ux = v$ has a unique solution x. Show that $v^{-1}u^{-1}$ is the inverse of uv.

Let G be the dihedral group $D(6)$. Find the solutions, x, y, of the equations $ax = ba^2$ and $a^{-1}ya = b$, expressing each of your solutions as one of the twelve elements written in the form explained in the paragraph before question 1. Do these equations have unique solutions? Find 2 solutions of the equation $xax^{-1} = a^5$. By squaring both sides, explain why the equation $xax^{-1} = a^3$ has no solution. Does $xax^{-1} = a^2$ have a solution?

3. Say what is meant by the statement that an element g of a finite group G has *order* k. State Lagrange's Theorem and use it to show that if a group G has an element g of order k, then k divides $|G|$. Deduce that if G has an element of order 2, then G has an even number of elements.

Say what is meant by saying that a group H is *cyclic* generated by g. Let g be an element of a group with g of order k. By considering the subgroup generated by g, show that for any divisor m of k, G has an element of order m.

Let n be an even integer and G be the dihedral group $D(n)$. Find a necessary and sufficient condition on n for G to have an element of order 4.

4. Show that if G is any group and H is a subgroup of G, then two (left) cosets xH and yH of H in G are either equal or disjoint. (You may assume that $xH = yH$ if and only if $y^{-1}x$ is an element of the subgroup H.) Say what is meant by *N is a normal subgroup of the group G*.

Let G be the dihedral group $D(4)$ written in the standard form explained before question 1. Let H be the subgroup with elements $\{1, b\}$. List the elements in the complete set of distinct left cosets of H in G and also list the elements in the distinct right cosets of H in G. Give an example of a left coset of H which is not a right coset. Is it possible to find representatives for the distinct left cosets of H in G such that these representatives form a subgroup of G?

Now let K be the subgroup with elements $\{1, a^2\}$. Explain why K is a normal subgroup of G and decide whether or not G/K is cyclic. Is it possible to find a subgroup L of G with $|L| = 4$ such that L and a^2L are the two distinct left cosets of L in G?

5. Let G be a finite group. Let g be an element of G. Define the conjugacy class of g and the centralizer, $C_G(g)$, of g in G. State a result which connects $|G|$, $|C_G(g)|$ and the number of elements in the conjugacy class of g.

Let G be the dihedral group $D(n)$ for n even, say $n = 2k$ (with $D(n)$ as defined in the first paragraph of this paper). Use the index laws to show that each of the n powers of a commutes with each other power of a and that the power a^k also commutes with b. Deduce that each power a^i (for $0 < i < k$) has 2 conjugates, but that 1_G and a^k each have one conjugate. By constructing representatives for the distinct left cosets of the subgroup $\{1, b, a^k, ba^k\}$, or otherwise, find the conjugates of b. Deduce that G has $(n+6)/2 = k+3$ conjugacy classes.

6. Let θ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that θ is a *homomorphism*. Define the kernel and the image of θ. State the homomorphism theorem.

Let G be the set of 4×4 matrices of the form

$$X = \begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

where a, b, c, d are real numbers, with a non-zero. You may assume that G is a subgroup of the set of all invertible 4×4 matrices under matrix multiplication. Decide which of the following maps $G \to H$ are homomorphisms, calculating the kernel and image of those maps which are homomorphisms:

(a) H is the set of real numbers under addition and the map θ_1 is given by $\theta_1(X) = b$,

(b) H is the set of non-zero real numbers under multiplication and $\theta_2(X) = a$.

Deduce that G has a normal abelian subgroup N with G/N abelian.

7. State the Sylow theorems and deduce that a group G has a unique Sylow p-subgroup if and only if the Sylow p-subgroups of G are normal.

Prove that a group with 15 elements is cyclic.

Let G be a group with 12 $(= 4 \times 3)$ elements. Show that G either has a normal Sylow 2-subgroup or a normal Sylow 3-subgroup.

Now let G denote the alternating group $A(4)$ consisting of the 12 even permutations on 4 letters. List the elements of $A(4)$ and determine the order of each of these elements. Hence calculate, for each prime p dividing $|G|$, the number of Sylow p-subgroups in G.

8. State the Jordan-Hölder Theorem explaining the terms you use.

Let H, K be subgroups of a group with K a normal subgroup of H and H/K having a prime number of elements. Prove that there is no normal subgroup L of H with $K < L < H$.

Find composition series for each of the following, justifying any assertions you make:

(1) a cyclic group with 6 elements,

(2) the dihedral group $D(2p)$ with $4p$ elements, where p is an odd prime,

(3) a group with 21 elements,

(4) the symmetric group $S(3)$.

[Hint: you will need Sylow theory in (3) and possibly in (2)]