## 2MP63 1999 Solutions

1. (a) A group is a set $G$ with a law of composition satisfying the following axioms:

(G1) for any $x, y \in G$, $xy$ is in $G$;

(G2) for any $x, y, z$ in $G$, $x(yz) = (xy)z$;

(G3) there is an element 1 in $G$ such that for all $g \in G$,

$$g1 = g = 1g.$$

(G4) given an element $g \in G$, there is an element $g^{-1}$ of $G$ with

$$gg^{-1} = 1 = g^{-1}g.$$

[4 marks]

The inverse of $X$ is $X$ itself and the inverse of $Y$ is the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

[2 marks]

Since $X = X^{-1}$, $X^2 = I$. Also, we note that $Y^2 = Y^{-1}$, so $Y$ has order 3.

[2 marks]

Thus it is clear that $\langle X \rangle$ contains $I, Y, Y^2, X, XY, XY^2$. To show that these six matrices form a group, we compute their multiplication table:

| | $I$ | $Y$ | $Y^2$ | $X$ | $XY$ | $X^2Y$ |
|---|---|---|---|---|---|---|
| $I$ | $I$ | $Y$ | $Y^2$ | $X$ | $XY$ | $X^2Y$ |
| $Y$ | $Y$ | $Y^2$ | $I$ | $XY^2$ | $X$ | $XY$ |
| $Y^2$ | $Y^2$ | $I$ | $Y$ | $XY$ | $XY^2$ | $Y$ |
| $X$ | $X$ | $XY$ | $XY^2$ | $I$ | $Y^2$ | $Y$ |
| $XY$ | $XY$ | $XY^2$ | $X$ | $Y$ | $I$ | $Y^2$ |
| $XY^2$ | $XY^2$ | $X$ | $XY$ | $Y^2$ | $Y$ | $I$ |

[6 marks]

This group is non-abelian since $XY$ and $YX$ are unequal ([1 mark]).

Let

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the required matrix. Then the condition that $XZ = ZX$ yields the matrix equation

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

so that $a = d$ and $b = c$. Then the condition that $YZ = ZY$ gives that

$$\begin{pmatrix} -a-b & -a-b \\ a & b \end{pmatrix} = \begin{pmatrix} b-a & -a \\ a-b & -b \end{pmatrix}.$$

Thus $b = 0$, so $Z$ has the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

The two matrices of determinant 1 of this form are $\pm I$, so the only one not in $G$ is $-I$. ([5 marks]).

2. Lagrange's Theorem states that if $|H|$ is a subgroup of a finite group $G$ then $|H|$ divides $|G|$ and $|G|/|H|$ is equal to the number of distinct cosets of $H$ in $G$ ([2 marks]). If $G$ has order $p$, let $x$ be any non-trivial element of $G$, then $|\langle x \rangle|$ has order dividing $p$. Since this order is not 1 by choice, it must be $p$, so $G = \langle x \rangle$ and so $G$ is cyclic ([2 marks]).

Now, we are given that $yx = x^{-1}y$ (the anchor step), so suppose that $yx^k = x^{-k}y$ then

$$yx^{k+1} = yx^k x = x^{-k}yx = x^{-(k+1)}y,$$

as required ([2 marks]).

To find the order of each of the 10 elements of $G$ we note that $x$ has order 5, so each power of $x$ has order 5. Also $yx^i yx^i = y(yx^{-i})x^i = y^2 = 1$, so each other element of $G$ has order 2. ([4 marks]).

Since $G$ has 10 elements, the possible orders of subgroups of $G$ are $1, 2, 5$ or 10. It follows that a proper subgroup of $G$ has prime order so is cyclic.

[3 marks]

To determine the subgroups with 2 elements, note that these are of the form $\{1, g\}$ where $g^2 = 1$, so $g$ is one of the five elements $yx^i$. There is only one subgroup with 5 elements ($\langle x \rangle$), so $G$ has 6 non-trivial proper subgroups ([4 marks]).

If now $H$ and $K$ are distinct proper subgroups of $G$, both $H$ and $K$ are cyclic of prime order and since $H \neq K$ we have that $H \cap K < H$ so $H \cap K = \{1\}$.

[3 marks]

3. Given groups $G$ and $H$, then $G \times H$ is the set of ordered pairs $(g, h)$ with $g \in G$ and $h \in H$, with group composition

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2).$$

[1 mark]

To see that this is a group check axioms:
(G1) is clear since $G$ and $H$ are groups;

(G2) just needs to be checked but follows directly from associativity in $G$ and $H$

$$(g_1, h_1)((g_2, h_2)(g_3, h_3)) = (g_1, h_1)(g_2g_3, h_2h_3) = (g_1(g_2g_3), h_1(h_2h_3))$$
$$= ((g_1g_2)g_3, (h_1h_2)h_3)) = ((g_1g_2, h_1h_2)(g_3, h_3) = ((g_1, h_1)(g_2, h_2))(g_3, h_3);$$

as required.

(G3) the identity is $(1_G, 1_H)$;

(G4) the inverse of $(g, h)$ is $(g^{-1}, h^{-1})$ ([4 marks]).

Now suppose that $G$ is abelian so that $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$ and also that $H$ is abelian $h_1h_2 = h_2h_1$ for all $h_1, h_2 \in H$. then

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2) = (g_2g_1, h_2h_1) = (g_2, h_2)(g_1, h_1)$$

so that $G \times H$ is abelian ([2 marks]).

For the converse, suppose that $G \times H$ is abelian so that

$$(g_1, h_1)(g_2, h_2) = (g_2, h_2)(g_1, h_1)$$

it then follows from the rule of composition that

$$(g_1g_2, h_1h_2) = (g_2g_1, h_2h_1)$$

so that $g_1g_2 = g_2g_1$ and $h_1h_2 = h_2h_1$, so that $G$ and $H$ are abelian. ([2 marks]).

The elements of $K$ are as follows: $(1, 1); ((1\ 2), x); (1, x^2);$ $((1\ 2); x^3);$ $(1, x^4)$ and $((1\ 2), x^5)$. ([2 marks]).

The distinct left cosets of $K$ in $G$ are therefore

$$\begin{array}{rcl} K &=& \{(1,1); ((1\ 2), x); (1, x^2); ((1\ 2), x^3); (1, x^4); ((1\ 2), x^5)\}; \\ (1, x)K &=& \{(1, x); ((1\ 2), x^2); (1, x^3); ((1\ 2), x^4); (1, x^5), ((1\ 2), 1)\}; \\ ((1\ 3), 1)K &=& \{((1\ 3), 1); ((1\ 2\ 3), x)); ((1\ 3), x^2); ((1\ 2\ 3), x^3); ((1\ 3), x^4); ((1\ 2\ 3), x^5)\} \\ ((1\ 3), x)K &=& \{((1\ 3), x); ((1\ 2\ 3), x^2); ((1\ 3), x^3); ((1\ 2\ 3), x^4); ((1\ 3), x^5); ((1\ 2\ 3), 1)\} \\ ((2\ 3), 1)K &=& \{((2\ 3), 1); ((1\ 3\ 2), x); ((2\ 3), x^2); ((1\ 3\ 2), x^3); ((2\ 3), x^4); ((1\ 3\ 2), x^5)\} \\ ((2\ 3), x)K &=& \{((2\ 3), x); ((1\ 3\ 2), x^2); ((2\ 3), x^3); ((1\ 3\ 2), x^4); ((2\ 3), x^5); ((1\ 3\ 2), 1)\} \end{array}$$

(Write completely for full (6) marks).

This is not the same as the decomposition into right cosets because

$$K((1\ 3), x) = ((1\ 3), 1); ((1\ 3\ 2), x); ((1\ 3), x^2); ((1\ 3\ 2), x^3); ((1\ 3), x^4); ((1\ 3\ 2), x^4)$$

and this is not a left coset. ([3 marks]).

4. Let $\vartheta : (G, \circ) \to (H, *)$ be a group homomorphism. Then for all $x, y$ in $G$, $\vartheta(x \circ y) = \vartheta(x) * \vartheta(y)$ ([1 mark]).

It follows that $\vartheta(1_G)\vartheta(g) = \vartheta(g)$ for all $g \in G$, so $\vartheta(1_G)$ is the identity element of $H$ (by uniqueness) as required.

Also $\vartheta(g)\vartheta(g^{-1}) = \vartheta(1_G) = 1_H$, so $\vartheta(g^{-1}) = \vartheta(g)^{-1}$ ([2 marks]).

We have

$$\ker\vartheta = \{g \in G : \vartheta(g) = 1_H\}$$

[1 mark]

and

$$\mathrm{im}\vartheta = \{h \in H : h = \vartheta(x) \text{ for some } x \in G\}.$$

[1 mark]

Then K=ker$\vartheta$ is a subgroup, because $1_G \in K$. If $x, y$ are in $K$, then $\vartheta(x) = \vartheta(y) = 1_H$, so $\vartheta(xy) = \vartheta(x)\vartheta(y) = 1_H 1_H = 1_H$, so $xy \in K$. Finally since $\vartheta(g^{-1}) = \vartheta(g)^{-1}$, $\vartheta(g^{-1}) = {1_H}^{-1} = 1_H$ and $g^{-1} \in K$. It only remains to show that $K$ is a normal subgroup. If $g \in G$ and $k \in K$ then

$$\vartheta(gkg^{-1}) = \vartheta(g) 1_H \vartheta(g)^{-1} = 1_H$$

so $gkg^{-1} \in K$ ([4 marks]).

The homomorphism theorem says

(a) im $\vartheta$ is a subgroup of $H$;

(b) ker $\vartheta$ is a normal subgroup of $G$;

(c) the quotient group $G/\ker\vartheta$ is isomorphic to im $\vartheta$ ([3 marks]).

Now the given $G$ is a subgroup because the product of two matrices with determinant $\pm 1, \pm i$ has determinant $\pm 1, \pm i$. Similarly the inverse of a matrix with one of these four determinants has determinant $\pm 1$ or $\pm i$. ([2 marks]).

Consider the map $\phi : G \to \mathbf{C}^{\times}$ defined by $\phi(X) = \det X$, then the kernel of this map is a normal subgroup of index 4 (since 4 possible determinants are allowed. ([4 marks])

Since $G/N$ is isomorphic to the cyclic group generated by $i$, $G/N$ is cyclic.([2 marks ]).

5. To show $G_X$ is a subgroup, note that the identity permutation is in $G_X$; also if $\pi$ and $\rho$ are in $G_X$, then $\pi(x) = \rho(x) = x$ for all $x \in X$, so

$$\pi(\rho(x)) = \pi(x) = x$$

for all $x \in X$, so that $\pi\rho$ is in $G_X$. Also, if $\pi$ is in $G_X$, then $\pi(x) = x$ for all $x \in X$. So $x = \pi^{-1}(x)$ for all $x \in X$ thus $G_X$ is a subgroup as required.

([3 marks])

The elements of $S(4)$ in $S(3)$ are $\{1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3)$ and $(1\ 3\ 2)\}$.

([1 mark]).

This is a subgroup with six elements, so its subgroups have order 1, 2, 3 or 6. Subgroups of order 1 or 6 are clear, so we need to find 4 subgroups of order 2 or 3. Such subgroups are cyclic so we just observe that $S(3)$ has three elements of order 2 (these being $(1\ 2), (1\ 3), (2\ 3)$) and two of order three $(1\ 2\ 3)$ and $(1\ 3\ 2)$. Since $\langle(1\ 2\ 3)\rangle = \langle(1\ 3\ 2)\rangle$, we obtain the required list ([4 marks]).

As for normal subgroups, $\{1\}$ and $G$ always are, $\langle(1\ 2\ 3)\rangle$ has index two so is normal, but none of the others are normal since

$$(1\ 2\ 3)(1\ 2)(1\ 3\ 2) = (2\ 3); (1\ 2\ 3)(1\ 3)(1\ 3\ 2) = (1\ 2); (1\ 2\ 3)(2\ 3)(1\ 3\ 2) = (1\ 3).$$

([2 marks])

To decide whether there is a normal subgroup of $S(4)$ contained in $S(3)$, note that such a subgroup would need to be a normal subgroup of $S(3)$ and so would be one of the three just considered. The subgroup $\{1\}$ is excluded, so we only need consider $S(3)$ itself and $A(3)$, neither of which are normal because $(3\ 4)(1\ 2\ 3)(3\ 4) = (1\ 2\ 4)$. ([5 marks]).

To decide whether $G$ has a proper normal subgroup containing $S(3)$ we first observe that such a subgroup would need to have order divisible by 6 and dividing 24, so would have order 12 (the general fact referred to in the question). However, $S(4)$ has a unique normal subgroup with 12 elements, the alternating group $A(4)$ consisting of even permutations. Since $S(3)$ contains some odd permutations, it is clear that $S(3)$ is not contained in $A(4)$ and therefore not in any proper normal subgroup of $S(4)$.

[5 marks]

6. A set $X$ is a $G$-set if there is an action $\circ : G \times X \to X$ such that:

$1_G \circ x = x$ for all $x \in X$

$gh \circ x = g \circ (h \circ x)$ for all $g, h \in G$ and all $x \in X$.

[2 marks]

The stabilizer $G_x$ of $x \in X$ is

$$G_x = \{g \in G : g \circ x = x\}.$$

[1 mark]

The orbit $O_x$ is

$$O_x = \{y : y = g \circ x \text{ for some } g \in G\}.$$

[1 mark]

To show that the stabilizer $G_x$ of $x$ is a subgroup note that if $g, h$ are in $G_x$ then $g \circ x = x = h \circ x$. Thus

$$gh \circ x = g \circ (h \circ x) = g \circ x = x$$

so $gh \in G_x$ as required. Also $1_G \in G_x$ so $G_x$ is non-empty. Finally, if $g \in G_x$ then $g \circ x = x$ so $g^{-1}g \circ x = g^{-1} \circ x$. It follows that $g^{-1} \circ x = 1_G \circ x = x$, so $g^{-1} \in G_x$ ([3 marks]).

The orbit-stabilizer theorem says

$G_x$ is a subgroup of $G$.

If $G$ is finite, then $|O_x| = |G : G_x|$.

[2 marks]

An example of a polynomial which has only itself in its orbit is $x_1+x_2+x_3+x_4$ ([2 marks]).

The polynomial $x_1x_2$ is stabilized by (1 2), by (3 4), so its stabilizer has at least four elements giving at most 6 elements in its orbit. However, the following are in the orbit, so must be the complete orbit:

$$x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4$$

([4 marks]).

Now consider $x_1x_2 + x_3x_4$. It is clear that the four permutations we found in the first part $\{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ all stabilize our polynomial. However, when we apply the 4-cycle (1 3 2 4) to our polynomial we see that it is also fixed by this permutation so the stabilzer has eight elements. We try to list three polynomials in its orbit, and easily obtain

$$x_1x_2 + x_3x_4, x_2x_3 + x_1x_4, x_3x_1 + x_2x_4$$

thus completing the determination ([5 marks]).

7. Let $p$ be a prime and $G$ be a finite group of order $p^k n$ where $p$ does not divide $n$. Then:

(1) $G$ has Sylow $p$-subgroups (subgroups of order $p^k$);

(2) the number of these is congruent to 1 mod $p$;

(3) if $P$ is a Sylow $p$-subgroup and $Q$ is any $p$-subgroup, there is an element $g$ of $G$ such that $gQg^{-1} \subseteq P$;

(4) any two Sylow $p$-subgroups are conjugate, the number of these divides $|G|$.

[4 marks]

Suppose that $G$ is a group of order 35=5 $\times$ 7 the number of Sylow 5-subgroups is $1, 6, 11, 16, 21, \ldots$ and divides 35, so is 1. The number of Sylow 7 subgroups is $1, 8, 15, 22, \ldots$ and divides 35 so is also 1. Thus $G$ has a unique Sylow 5-subgroup,

$P$, say, and a unique Sylow 7-subgroup $Q$, say. These are each normal with $P$ containing all 4 non-identity elements of $G$ of order 5 and $Q$ containing all 6 non-identity elements of $G$ of order 7. It follows by Lagrange that there must be elements of $G$ of order 35 (the only other divisor of 35), so $G$ is cyclic. ([5 marks]).

Now suppose that $G$ is a group with $105{=}3 \times 5 \times 7$. The number of Sylow 3-subgroups is either 1 or 7. The number of Sylow 5-subgroups is either 1 or 21 and the number of Sylow 7-subgroups is 1 or 15. Suppose $G$ has more than 1 (and so 15) sylow 7-subgroups. These 15 distinct subgroups would all intersect in the identity element, giving in total 90 elements of order 7, and only leaving 15 elements of $G$ to be distributed over the Sylow 3 and 5 subgroups. It would follow that there could only be one of each. Now consider two cases (a) $G$ has a normal Sylow 7-subgroup $P$. Then $G/P$ would have order 15 and so would be cyclic. By the correspondence theorem, the lift of a Sylow 5-subgroup of this quotient back to $G$ would give a normal subgroup of order 35. In case (b), we have seen that $G$ has a normal Sylow 5-subgroup $Q$, so that $G/Q$ has order 21. Since a group of order 21 has a normal Sylow 7-subgroup, we can apply the correspondence theorem again to still obtain a normal subgroup of order 35. ([7 marks])

Finally, suppose $G$ has $56 = 2^3 \times 7$ elements, but does not have a unique Sylow 7 subgroup, so that the number of Sylow 7-subgroups is 8. These eight subgroups intersect pairwise in $\{1\}$, giving 48 elements of order 7 and only leaving room for one (and therefore normal) Sylow 2-subgroup. ([4 marks]).

8. The Jordan-Hölder Theorem says that any two composition series of a group are isomorphic ([1 mark]). A composition series is a finite series of subgroups, each normal in the next

$$G = G_0 \geq G_1 \geq \cdots G_k = \{1\}$$

which can not be refined without repeating terms ([1 mark]). Two composition series are isomorphic if there is a bijection between the quotient groups in the respective series so that corresponding quotient groups are isomorphic ([1 mark]).

(a) Let $G$ be a cyclic group of order 4 generated by $x$ (so $x^4 = 1$). Then $\langle x^2 \rangle$ is a subgroup of $G$ which is normal since $G$ is abelian. It follows (since 2 is prime) that a composition series for $G$ is

$$G \geq \langle x^2 \rangle \geq \{1\}.$$

[3 marks]

(b) Now let $G$ be a non-cyclic of order 4 and let $y$ be a non-identity element of $G$ (so that $y^2 = 1$). Apply the same argument as in (1) with $\langle y \rangle$ replacing $\langle x^2 \rangle$, to obtain the composition series

$$G \geq \langle y \rangle \geq \{1\}.$$

($\langle y \rangle$ is normal since it has index 2).

[3 marks]

(c) Next, let $G$ be cyclic of order 6 (so it is generated by $x$ with $x^6 = 1$). Consider the subgroup $\langle x^2 \rangle$ of order 3. It is normal because $G$ is abelian. The series

$$G \geq \langle x^2 \rangle \geq \{1\}$$

cannot be refined beause 2 and 3 are primes, so is a composition series.

[3 marks]

(d) Now let $G$ be the alternating group $A(4)$. The four elements

$$1; (1\ 2)(3\ 4); (1\ 3)(2\ 4); (1\ 4)(2\ 3)$$

form a subgroup $V$ which is normal since the three non-idenitity elements form a conjugacy class. So we have a series for $G$

$$G \geq V \geq \{1\}$$

since $G/V$ has order 3 this bit cannot be refined, so we are left with the problem of whether $V$ has a better composition series. This is solved in (b), so a composition series is

$$G \geq V \geq \{1, (1\ 2)(3\ 4)\} \geq \{1\}$$

[5 marks])

(e) We finally turn to the dihedral group $D(4)$. The subgroup $\langle x \rangle$ is cyclic of order 4 and is normal because it is of index 2. Also $\langle x^2 \rangle$ is a subgroup of this and is normal because $\langle x \rangle$ is abelian, so a composition series is

$$G \geq \langle x \rangle \geq \langle x^2 \rangle \geq \{1\}.$$

[3 marks]