Througout this paper, $D(n)$ denotes the dihedral group of symmetries of a regular n-sided polygon. Thus $D(n)$ has $2n$ distinct elements and may be generated by an element a of order n together with an element b of order 2, with the relation $ab = ba^{-1}$. Then the distinct elements of $D(n)$ are the n powers of a together with n elements of the form b times a power of a. You may also assume throughout that for $0 \le i \le n-1$, we have $b^{-1}a^i b = a^{-i}$.

Also $S(n)$ denotes the group of $n!$ permutations on n symbols and $A(n)$ is its subgroup consisting of even permutations.

1. Define a *group*. Let G be the group generated by the permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 5 & 6 & 7 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

Write down 8 permutations in G, and show that these 8 permutations form the group G. Find the order of each element of G, and show that there is a non-trivial element z in G such that $zg = gz$ for every element g of G.

2.

(1) Let X be a set with two elements. List the maps from X to itself. Is the set of these maps a group under composition of functions?

(2) Define what is meant by saying that H is a *cyclic* subgroup of a group G. State Lagrange's Theorem and use it to show that a group with a prime number of elements is cyclic.

Let G be the dihedral group $D(4)$. Give an example of a subgroup of G which has four elements and is cyclic and an example of a subgroup of G which has four elements and is not cyclic.

3. Show that if G is any group and H is a subgroup of G, then two (left) cosets xH and yH of H in G are equal if and only if $y^{-1}x$ is an element of the subgroup H.

Let G be the dihedral group $D(8)$ with 16 elements, in the notation explained before question 1. Let H be the set of elements $\{1, a^2, a^4, a^6\}$. Explain why H is a subgroup of G. Calculate the complete list of distinct left cosets of H in G and also the list of distinct right cosets of H in G. Deduce that H is a normal subgroup of G and decide whether or not G/H is cyclic. For any element g in G explain why g^2 is an element of the subgroup H.

4. Let G be a finite group. Let g be an element of G. Define the conjugacy class of g and the centralizer, $C_G(g)$, of g in G. Prove that, for any g in G, the centralizer $C_G(g)$ is a subgroup of G and that the number of distinct elements in the conjugacy class of g is equal to $|G|/|C_G(g)|$.

Let G be the dihedral group $D(n)$ for n odd (as defined in the first paragraph of this paper). Show that each of the n powers of a commutes with each other power of a. Deduce that each a^i has 2 conjugates. Find the number of conjugates of b. Deduce that G has $(n+3)/2$ conjugacy classes.

5. Let θ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that θ is a homomorphism. Show that if θ is a homomorphism then $\theta(1_G) = 1_H$. Show also that if g and h are elements of G with h being the inverse of g (with respect to the operation $\circ$), then $\theta(h)$ is the inverse of $\theta(g)$ (with respect to the operation $*$). Define the kernel and the image of θ. Prove that the kernel of θ is a normal subgroup of G. State the homomorphism theorem.

Let G be the set of 2×2 matrices of the form

$$X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where a, b, c are real numbers, with ac non-zero. You may assume that G is a subgroup of the set of all invertible 2×2 matrices under matrix multiplication. Decide which of the following maps $G \to H$ are homomorphisms, calculating the kernel and image of those maps which are homomorphisms:

(a) H is the set of real numbers under addition and the map θ_1 is given by $\theta_1(X) = b$,

(b) H is the set of non-zero real numbers under multiplication and $\theta_2(X) = a$.

6. Give rules which enable the sign of a permutation to be determined. Given a permutation π expressed as a product of disjoint cycles, explain how to calculate the order of π. Use your rules to calculate the order and sign of the permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 9 \end{pmatrix}.$$

Show that $A(n)$ is a normal subgroup of $S(n)$. Give the order of each element in $A(4)$.

Find the smallest n such that $S(n)$ has an element of order 6 and the smallest m such that $A(m)$ has an element of order six.

7. State the Sylow theorems and show that a group G has a unique Sylow p-subgroup if and only if the Sylow p-subgroups of G are normal.

Prove that a group with 33 elements is cyclic.

Let G be a group with 56 $(= 8 \times 7)$ elements. Show that G either has a normal Sylow 2-subgroup or a normal Sylow 7-subgroup.

Now let G denote the dihedral group $D(6)$. Using the notation before question 1, list the elements of $D(6)$ and determine the order of each of these elements. Hence calculate, for each prime p dividing $|G|$, the number of Sylow p-subgroups in G.

8. State the Jordan-Hölder Theorem explaining the terms you use.

Let H, K be subgroups of a group with K a normal subgroup of H and H/K having a prime number of elements. Prove that there is no normal subgroup L of H with $K < L < H$. Find composition series for each of the following, justifying any assertions you make:

(1) a cyclic group with 8 elements,

(2) the dihedral group $D(3)$,

(3) a group with 35 elements,

(4) the dihedral group $D(8)$.

[Hint: you will need Sylow theory in (3)]