1. Define a *group*. Let X and Y be the 2×2 matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Find the matrix inverse of each of X and Y. Find also the smallest integers r and s such that $X^r = Y^s = I$. Determine the group table for $G = \langle X, Y \rangle$ under matrix multiplication. Is G abelian? Find all those 2×2 invertible matrices Z satisfying the three conditions that $XZ = ZX$, $ZY = YZ$ and that Z is a matrix of determinant 1 not in G.

2. State Lagrange's Theorem and use it to show that a group G with p elements (where p is a prime) is cyclic.

Now let G be the dihedral group of symmetries of a regular 5-sided polygon. Thus

$$G = \{1, x, x^2, x^3, x^4, y, yx, yx^2, yx^3, yx^4\}$$

where x corresponds to rotation through 72 degrees and y corresponds to a reflection. You may assume that $yx = x^{-1}y$. Prove, by induction on i that $yx^i = x^{-i}y$. Use this fact to find the order of each element of G.

Use Lagrange's Theorem to list the number of elements in each possible subgroup of G and deduce (from you list of numbers) that every proper subgroup of G is cyclic.

Determine the complete list of the 8 distinct subgroups of G. Explain why it is true that if H, K are distinct proper subgroups of G then $H \cap K = \{1\}$.

3. Define the direct product $G \times H$ of G and H. Prove that the direct product $G \times H$ is a group which is abelian if and only if G and H are both abelian. Consider L, the direct product of groups $S(3)$ and H where H is a cyclic group of order 6 generated by an element x. List the elements in the subgroup K of L generated by the ordered pair $((1\ 2), x)$, and find the decomposition of L into distinct left cosets of K. Is this the same as the decomposition of L into right cosets of K?

4. Let ϑ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that ϑ is a homomorphism. Show that if ϑ is a homomorphism then $\vartheta(1_G) = 1_H$ and $\vartheta(g^{-1}) = \vartheta(g)^{-1}$. Define the kernel and the image of ϑ, and prove that the kernel of ϑ is a normal subgroup of G. State the homomorphism theorem.

Let G be the set of all those complex 2×2 matrices whose determinant is one of $1, i, -1$ or $-i$ (where $i^2 = -1$). Explain why G is a subgroup of the group of invertible 2×2 matrices under matrix multiplication. Show that G has a normal subgroup N of index 4. Decide whether G/N is cyclic or not.

5. Let G be the symmetric group $S(n)$ and X be any subset of $\{1, \ldots, n\}$. Let G_X be the set of elements of G which fix each number in X (so if $h \in G_X$ then $h(x) = x$ for all $x \in X$). Show that G_X is a subgroup of G. Now suppose that $n = 4$ and let $S(3)$ be the subgroup of $S(4)$ consisting of those elements in $S(4)$ which fix 4. List the elements in $S(3)$, and find 6 subgroups of $S(3)$. Which of these are normal subgroups of $S(3)$? Decide whether or not there is a non-trivial normal subgroup of $S(4)$ contained in $S(3)$ and also decide whether or not $S(3)$ is contained in a proper normal subgroup of $S(4)$. [You should state clearly any general results about the group $S(4)$ you use.]

6. Let G be a group. Define the terms *G-set*, *orbit* and *stabilizer*. Prove that the stabilizer is a subgroup of G and state the orbit-stabilizer theorem.

Let X be the set of all polynomials with real coefficients in the four variables x_1, x_2, x_3, x_4. For each permutation π in $S(4)$ and each polynomial f in X define the polynomial f^π to be that obtained from f by permuting (the subscripts on) the variables using the permutation π. Give an example of a polynomial in X which has only itself in its orbit under $S(4)$. Find the orbits and stabilizers of the following polynomials:

(1) $x_1 x_2$

(2) $x_1 x_2 + x_3 x_4$.

7. State the Sylow theorems. Prove the following

(1) a group with 35 elements is cyclic;

(2) a group with 105 elements has a normal subgroup of order 35;

(3) a group with 56 elements has either a normal Sylow 7-subgroup or a normal Sylow 2 subgroup.

8. State the Jordan-Hölder Theorem explaining the terms you use. Find composition series for each of the following:

(1) a cyclic group of order 4;

(2) a non-cyclic group of order 4;

(3) a cyclic group of order 6;

(4) the alternating group $A(4)$;

(5) the dihedral group $D(4)$.