Throughout this paper, $D(n)$ denotes the dihedral group of symmetries of a regular n-sided polygon. Thus $D(n)$ has $2n$ distinct elements and may be generated by an element a of order n together with an element b of order 2, with the relation $ab = ba^{-1}$. Then the distinct elements of $D(n)$ are the n powers of a together with n elements of the form b times a power of a.

Also $S(n)$ denotes the group of $n!$ permutations on n symbols and $A(n)$ is its subgroup consisting of even permutations.

1. Define a *group*. Let π be the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}.$$

Find another permutation ρ on $\{1, 2, 3, 4\}$ such that $\rho(1) = 1$ and $\pi\rho = \rho\pi^{-1}$. Now let G be the group generated by π and ρ. Write down 8 permutations in G. Show that these 8 permutations actually form the group G. Show that there is a non-identity element z in G such that $zg = gz$ for every element g in G. [20 marks]

2. Let u and v be elements in a group G. Prove, carefully using the group axioms, that the equation $ux = v$ has a unique solution x in G.

Let G be a cyclic group of order 30 generated by an element g. Solve the equation $x^5 = e$ in G. Solve the system of equations $x^{15} = e$ and $x^6 = e$ in G.

Let G be the dihedral group $D(10)$ with 20 elements, in the notation explained before question 1. Solve the equation $ba^8x = a^2$ in G and express your solution as one of the 20 elements written in the form explained before question 1. Find two solutions of the equation $ax = xa^{-1}$ and express each of your solutions as one of the 20 elements written in the form explained before question 1. Given elements u and v in G, does the equation $ux^5 = v$ have a unique solution? [20 marks]

3. Say what is meant by saying that H is a *subgroup* of a group G. State Lagrange's theorem. Say what is meant by saying that a subgroup H is *cyclic* generated by g. Let p be a prime number. Use Lagrange's theorem to show that a group with p elements is cyclic. Deduce that if H is a subgroup of G with p elements and K is a subgroup of G with q elements, where p and q are distinct prime numbers, then $H \cap K = \{e\}$.

Calculate the order of each element of the alternating group $A(4)$ of even permutations on 4 elements. Prove that every non-trivial cyclic subgroup of $A(4)$ has prime order. Deduce that if H and K are cyclic subgroups of $A(4)$, then either $H = K$ or $H \cap K = \{e\}$. Find all subgroups of the group $S(4)$ of permutations on 4 symbols, which contain all even permutations. [20 marks]

4. Show that two (left) cosets aH and bH of H in G are equal if and only if the element $a^{-1}b$ is in the subgroup H. Say what is meant by N is a *normal subgroup* of the group G.

Let G be the dihedral group $D(10)$ with 20 elements, in the notation explained before question 1. Explain why the subset

$$H = \{e, a^2, a^4, a^6, a^8\}$$

is a subgroup of G. Calculate the complete list of distinct left cosets of H in G and also the list of distinct right cosets of H in G. Deduce that H is a normal subgroup of G. Decide whether or not G/H is cyclic. For any element g of G explain why g^2 is an element of the subgroup H. [20 marks]

5. Let f be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that f is a *homomorphism.* Define the *kernel* and the *image* of f. State the homomorphism theorem. Let G be the set of invertible 2×2 matrices of the form

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are real numbers. You may assume that G is a subgroup of the group of all invertible 2×2 real matrices under matrix multiplication. Decide whether the following two maps $G \to H$ are homomorphisms, calculating the kernel and the image of those maps which are homomorphisms:

(1) H is the group of all real numbers under addition and f is given by $f(A) = a$,

(2) H is the group of non-zero real numbers under multiplication and h is given by $h(A) = a^2 - b^2$.

Deduce that G has a normal subgroup N with G/N abelian. [20 marks]

6. Give rules which enable the *sign* of a permutation to be determined. Given a permutation π expressed as a product of disjoint cycles, explain how to calculate the order of π. Use your rules to calculate the order and the sign of the permutations

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 \end{pmatrix}$$

and

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 4 & 2 & 6 & 3 & 8 & 5 & 9 & 7 & 1 \end{pmatrix}.$$

Show that the subset $A(n)$ of even permutations is a normal subgroup of $S(n)$. Is the set of odd permutations a subgroup of $S(n)$?

Show that a permutation of odd order in $S(n)$ is even. Give an example of an even permutation of order k for $k = 2, 3$. If π is any element in $S(n)$, show that π^2 is an element in $A(n)$. [20 marks]

7. State the Sylow theorem. Show that a group G has a unique Sylow p-subgroup if and only if G has a normal Sylow p-subgroup.

(1) Prove that a group with 35 elements is cyclic.

(2) Let G be a group with 56 elements. Show that G either has a normal Sylow 2-subgroup or a normal Sylow 7-subgroup.

(3) Now let G denote the symmetric group $S(4)$ of permutations on 4 symbols. Then calculate, for each prime p dividing the order of G, the number of Sylow p-subgroups in G. [20 marks]

8. State the Jordan Hölder Theorem explaining the terms you use. Let H and K be subgroups of a group G. Let K be a normal subgroup of H. We assume that H/K has a prime number of elements. Prove that there is no normal subgroup L of H with $K < L < H$ and $L \neq H$, $L \neq K$. Find the composition series for each of the following, justifying any assertions you make:

(1) a cyclic group with 4 elements,

(2) a non-cyclic group with 4 elements,

(3) a group with 21 elements,

(4) the symmetric group $S(3)$,

(5) the dihedral group $D(6)$.

[Hint: you will need Sylow theory in (3) and may use it elsewhere.] [20 marks]