THE UNIVERSITY
of LIVERPOOL

1. Define a *group*. Let X and Y be the 2×2 matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Find the matrix inverse of each of X and Y. Find also the order of X and Y. Determine the composition table for $G = \langle X, Y \rangle$ under matrix multiplication. Is G abelian? Find all those 2×2 invertible matrices Z satisfying the three conditions that $XZ = ZX, ZY = YZ$ and that Z is a matrix of determinant 1 not in G.

[20 marks]

2. Let u and v be elements in a group G. Prove, carefully using the group axioms, that the equation $ux = v$ has a unique solution x.

Let G be the dihedral group $D(4)$ with set of elements

$$G = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}.$$

Solve the equation $a^2x = ba$ for x and give the solution x as one of the listed elements of G. Calculate the square of each element of G. Find a solution of the equation $x^2 = a^2$. Is the solution unique? Explain why neither of the equations $x^2 = b$ and $bx = xa$ has a solution in G.

[20 marks]

3. State Lagrange's theorem and use it to show that a group G with prime order, $p = |G|$, is cyclic.

Now let G be the dihedral group of symmetries of a regular 5-gon. Thus

$$G = \{1, x, x^2, x^3, x^4, y, yx, yx^2, yx^3, yx^4\}$$

where x corresponds to a rotation through 72 degrees and y corresponds to a reflection. You may assume that $yx = x^{-1}y$. Prove by induction on i, that $yx^i = x^{-i}y$. Use this fact to find the order of each element of G.

Use Lagrange's theorem to list the number of elements in each possible subgroup of G and deduce that every proper subgroup of G is cyclic.

Determine the complete list of the 8 distinct subgroups of G. Explain why it is true that if H, K are distinct proper subgroups of G, then $H \cap K = \{1\}$.

[20 marks]

4. Show that if G is any group and H is a subgroup of G, then the two left cosets xH and yH in G are equal if and only if $y^{-1}x \in H$.

Let G be the dihedral group with 12 elements, the group of symmetries of the regular hexagon, so that G has elements x of order 6 and y of order 2 with $xy = yx^{-1}$. You may assume that G is the set of elements

$$G = \{1, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}.$$

Let H be the set of elements $\{1, x^2, x^4\}$. Check that H is a subgroup of G and calculate the complete list of left and of right cosets of H in G. Deduce that H is a normal subgroup of G and decide whether or not G/H is cyclic. Give an example of a subgroup K of G with $|K| = 6$, such that K does not contain x^3. Deduce that G has two proper subgroups K and L with $K \cap L = \{1\}$ and $G = KL$.

[20 marks]

5. Let θ be a map between groups $G, \circ$ and $H, *$. State the conditions for θ to be a *homomorphism*. Show that if θ is a homomorphism, then $\theta(1_G) = 1_H$. Define the *kernel* and the *image* of θ. Prove that the kernel of θ is a normal subgroup of G. State the homomorphism theorem.

Let G be the set of 2×2 matrices of the form

$$X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where a, b, c are real numbers, with ac non-zero. Assume that G is a subgroup of the group of invertible 2 matrices under matrix multiplication. Decide which of the following maps $G \to H$ are homomorphisms, and calculate the kernel and image in case the map is a homomorphism:

1. H is the group of real numbers under addition and θ_1 is given by $\theta_1(X) = b$.
2. H is the group of non-zero real numbers under multiplication and θ_2 is given by $\theta_2(X) = a$.

[20 marks]

6. Give rules by which the sign of a permutation can be determined, and use these to calculate the signs of the permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

Prove that the set $A(n)$ of even permutations forms a normal subgroup of the symmetric group $S(n)$. If π is any permutation on n symbols, show that π^2 is in $A(n)$. Give an example of an even permutation of order 2, and show that every permutation of odd order is an even permutation. Can every cycle be written as a product of 3-cycles?

[20 marks]

7. State the Sylow theorems and show that a group G has a unique Sylow p-subgroup if and only if the Sylow p-subgroups of G are normal.

1. Prove that a group of order 33 is cyclic.
2. Prove that a group of order 12 has either a normal Sylow 2-subgroup or a normal Sylow 3-subgroup.
3. Let G now denote the dihedral group $D(6)$ as in question 4. Determine the order of each element. Then calculate, for each prime dividing $|G|$, the number of Sylow p-subgroups in G.

[20 marks]

8. State the Jordan Hölder Theorem explaining the terms you use.
Find the composition series for each of the following, justifying any assertions you make:

1. a cyclic group with 6 elements,
2. the dihedral group $D(9)$
3. a group with 22 elements,
4. the symmetric group $S(3)$.

[Hint: you will need Sylow theory in (3) and may use it elsewhere]

Give an example of a group with two different composition series.

[20 marks]