Througout this paper, $D(n)$ denotes the dihedral group of symmetries of a regular n-sided polygon. Thus $D(n)$ has $2n$ distinct elements and may be generated by an element a of order n together with an element b of order 2, with the relation $ab = ba^{-1}$. Then the distinct elements of $D(n)$ are the n powers of a together with n elements of the form b times a power of a. You may also assume throughout that for $0 \leq i \leq n-1$, we have $a^i b = ba^{-i}$.

1. Define a *group.* Let G be the group generated by the permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

Write down 8 permutations in G, and show that these 8 permutations form the group G. Find the order of each element of G, and show that there is a non-trivial element z in G such that $zg = gz$ for every element g of G.

2. Let u and v be elements of a group G. Prove that the equation $ux = v$ has a unique solution x.

Let G be the dihedral group $D(4)$. Write the solution, x, of the equation $a^2x = ba$ as one of these eight elements written in the form explained in the paragraph before question 1. Calculate the square of each element of G. Find a solution of the equation $x^2 = a^2$. Does this equation have a unique solution? Explain why neither of the equations $x^2 = b$ and $bx = xa$ has a solution in G.

3. Define the terms *subgroup* and *cyclic subgroup.* State Lagrange's Theorem. Let p be a prime number. Prove that a group with p elements is cyclic.

Calculate the orders of each element of (a) the dihedral group $D(6)$ and (b) the alternating group (group of even permutations on 4 symbols) $A(4)$. Give examples of:

1 A group with 12 elements with a cyclic subgroup H for which $|H|$ is not a prime number,

2 A group with 12 elements in which every cyclic subgroup has prime order.

4. Show that if G is any group and H is a subgroup of G, then two (left) cosets xH and yH of H in G are either equal or disjoint. (You may assume that $xH = yH$ if and only if $y^{-1}x$ is an element of the subgroup H.) Define what is meant by saying that N is a *normal subgroup* of the group G.

Let G be the dihedral group $D(3)$ written in the standard form explained before question 1. Let H be the subgroup with elements $\{1, b\}$. Calculate the complete list of distinct left cosets of H in G and also the list of distinct right cosets of H in G. Give an example of a left coset of H which is not a right coset.

Now let K be the subgroup with elements $\{1, a, a^2\}$. Explain why K is a normal subgroup of G and decide whether or not G/K is cyclic.

Let $\mathbf{Z}$ be the group of integers under addition, and n be a positive integer. Show that the set N, of multiples of n, is a subgroup of $\mathbf{Z}$ and explain why N is a normal subgroup of $\mathbf{Z}$. When $n = 10$ find the distinct (left) cosets of N in $\mathbf{Z}$.

5. Let ϑ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that ϑ is a homomorphism. Show that if ϑ is a homomorphism then $\vartheta(1_G) = 1_H$. Show also that if g and h are elements of G with h being the inverse of g (with respect to the operation $\circ$), then $\vartheta(h)$ is the inverse of $\vartheta(g)$ (with respect to the operation $*$). Define the kernel and the image of ϑ, and state the homomorphism theorem.

Let G be the group of 3×3 matrices of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix},$$

(where a_1, a_2, a_3 are real numbers with a_1 non-zero) under matrix multiplication. In each of the following cases, decide whether the given map is a homomorphism or not. When the map is a homomorphism, calculate its kernel and its image.

(a) Let θ_1 be the map from G into the multiplicative group of non-zero real numbers defined by $\theta_1(A) = a_1$;

(b) Let θ_2 be the map from G into the additive group of real numbers given by $\theta_2(A) = a_2$;

(c) Let θ_3 be the map from G into the additive group of real numbers defined by $\theta_3(A) = a_3$.

Deduce that G has an abelian normal subgroup N with G/N abelian.

6. Let G be a group. Let g be an element of G. Define the conjugacy class of g and the centralizer of g in G. Prove that, for any g in G, the centralizer $C_G(g)$ is a subgroup of G. State a result which connects $|G|$, $|C_G(g)|$ and the number of elements in the conjugacy class of g.

Let G be the group $D(4)$ of symmetries of a square. Find the conjugacy class of (i) a^2 and (ii) b.

Prove, by induction on k, that for all x and g in G, and all positive integers k, $(x^{-1}gx)^k = x^{-1}g^kx$. Deduce that g and $x^{-1}gx$ have the same order.

Now let G be a group in which only one element, z say, has order 2. Show that for all g in G, $gz = zg$.

7. State the Sylow theorems and show that a group G has a unique Sylow p-subgroup if and only if the Sylow p-subgroups of G are normal.

Prove that a group with 15 elements is cyclic. Let G be a group with 80 ($=16 \times 5$) elements. Show that G either has a normal Sylow 2-subgroup or a normal Sylow 5-subgroup,

Now let G denote the symmetric group $S(4)$ of permutations on four symbols. For each prime p dividing $|G|$, calculate the number of Sylow p-subgroups in G.

8. State the Jordan-Hölder Theorem explaining the terms you use.

Let H, K be subgroups of a group with K a normal subgroup of H and H/K having a prime number of elements. Prove that there is no normal subgroup L of H with $K < L < H$. Find composition series for each of the following, justifying any assertions you make:

(1) a cyclic group with 10 elements,

(2) the dihedral group $D(4)$,

(3) a group with 39 elements,

(4) the symmetric group $S(3)$.

[Hint: you will need Sylow theory in (3)]