1. Define a *group.* Let X be the set of 2×2 matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b and c are real numbers. Explain why X is **not** a group under matrix multiplication. Let G be the subset of X consisting of those matrices A with $ac \neq 0$. Prove that G is a group under matrix multiplication (you may assume that this operation is associative). Find all elements of G of order 2, and find two matrices A, B each of order 2 such that AB has infinite order. Find two matrices C, D in G with $CD \neq DC$.

2. Define a *subgroup* of a group. State Lagrange's Theorem. Let p be a prime number. Prove that a group with p elements is cyclic. If H and K are subgroups of a group G, show that $H \cap K$ is a subgroup of H and also a subgroup of K. Deduce that if H, K are subgroups of G with H having p elements and K having q elements (where p and q are distinct prime numbers) then $H \cap K = \{1\}$. Show it is also the case that if H and K are distinct subgroups each with p elements then $H \cap K = \{1\}$.

Now let G be a group with 10 elements and H, K be distinct subgroups neither of which is equal to G. Prove that $H \cap K = \{1\}$.

3. Show that if G is any group and H is a subgroup of G, then two (left) cosets xH and yH in G are equal if and only if $y^{-1}x$ is an element of the subgroup H.

Let G be the dihedral group with 12 elements (the group of symmetries of a regular hexagon), so that G has elements x of order 6 and y of order 2 with $xy = yx^{-1}$. You may assume that the elements of G are $\{1, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$. Let H be the set of elements $\{1, x^2, x^4\}$. Check that H is a subgroup of G and calculate the complete lists of left and of right cosets of H in G. Deduce that H is a normal subgroup of G and decide whether or not G/H is cyclic. Give an example of a subgroup K of G with $|K| = 6$ such that K does not contain x^3. Deduce that G has two proper normal subgroups A, B with $A \cap B = \{1\}$ and $G = AB$.

4. Let ϑ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that ϑ is a homomorphism. Show that if ϑ is a homomorphism then $\vartheta(1_G) = 1_H$. Show also that if g and h are elements of G with h being the inverse of g (with respect to the operation $\circ$), then $\vartheta(h)$ is the inverse of $\vartheta(g)$ (with respect to the operation $*$). Define the kernel and the image of ϑ. State the homomorphism theorem.

Let G be the set of 2×2 matrices of the form

$$X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a, b are real numbers, not both zero. You may assume that G is a subgroup of the set of all invertible 2×2 matrices under matrix multiplication. Decide which of the following three maps $G \to H$ are homomorphisms, calculating the kernel and image of those maps which are homomorphisms:

(a) H is the set of real numbers under addition and the map θ_1 is given by $\theta_1(X) = a + b$,

(b) H is the set of non-zero real numbers under multiplication and $\theta_2(X) = a^2 + b^2$,

(c) H is the set of non-zero complex numbers $a + ib$ (where $i^2 = -1$) under multiplication and $\theta_3(X) = a + ib$.

5. Give rules which enable the sign of a permutation to be determined, and use these to calculate the signs of the permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

Prove that the set $A(n)$ of even permutations (sign 1) forms a normal subgroup of the symmetric group $S(n)$. If π is any permutation on n symbols, show that π^2 is in $A(n)$. Give an example of an even permutation of order 2, and show that every permutation of odd order is an even permutation. Can every cycle be written as a product of 3-cycles?

6. Let G be a group. Define the terms conjugacy class and centralizer $C_G(g)$ of an element g in G. Prove that $C_G(g)$ is a subgroup of G and that the number of distinct conjugates of g is equal to the index of $C_G(g)$ in G.

Determine the complete set of conjugacy classes for both the dihedral group with 8 elements and the dihedral group with 10 elements.

7. State the Sylow theorems.

Prove that a group with 35 elements is cyclic, and explain why a similar proof does not establish that a group with 14 elements is cyclic.

Prove that a group with 12 elements either has a normal Sylow 2-subgroup or a normal Sylow 3-subgroup.

Let p be an odd prime and G be the dihedral group $D(p)$ with $2p$ elements, so that G has two generators a, b with $a^p = 1 = b^2$ and $b^{-1}ab = a^{-1}$. Prove, by induction, that for all positive integers i, $b^{-1}a^ib = a^{-i}$. Calculate the number of Sylow subgroups for each prime dividing the order of G.

8. State the Jordan-Hölder Theorem explaining the terms you use. Find composition series for each of the following, justifying any assertions you make:

(1) a cyclic group of order 6;

(2) a non-cyclic group of order 4;

(3) a group of order 21;

(4) the symmetric group $S(4)$.

[Hint: you will need to use Sylow theory in (3)].

Give an example of a group with two different composition series.