1. Define a *group*. Let G be the set of 3×3 matrices of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix},$$

where a, b and c are real numbers. Find the matrix inverse of A. Prove that G is a group under matrix multiplication (you may assume that this multiplication is associative). Show that G has no elements of order 2.

Let Z denote the matrix

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Find the order of Z and show that $ZA = AZ$ for every element A of G.

Give an example of two elements A, B of G with $AB \neq BA$.

2. Let u and v be elements of a group G. Prove that the equation $ux = v$ has a unique solution.

Let G be the dihedral group $D(4)$ so that G has generators a and b with $a^4 = 1 = b^2$ and $ab = ba^{-1}$. You may assume that the 8 elements of G are $1, a, a^2, a^3$ and b, ba, ba^2, ba^3. Write the solution, x, of the equation $bax = a^2$ as one of these eight elements. Calculate the square of each element of G. Show that the equation $bax^2 = a^2$ does not have a solution in G. Given u, v in G, does the equation $ux^3 = v$ have a unique solution?

3. Show that if G is any group and H is a subgroup of G, then two (left) cosets xH and yH of H in G are equal if and only if $y^{-1}x$ is an element of the subgroup H.

Let G be the dihedral group with 12 elements (the group of symmetries of a regular hexagon), so that G has an element x of order 6, an element y of order 2 and $xy = yx^{-1}$. You may assume that the elements of G are $\{1, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$. Let H be the subgroup with elements $\{1, x^3\}$. Calculate the complete list of distinct left cosets of H in G and also the list of distinct right cosets of H in G. Deduce that H is a normal subgroup of G and decide whether or not G/H is cyclic. Give an example of a subgroup K of G with $|K| = 2$ such that K is not a normal subgroup of G.

4. Let ϑ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that ϑ is a homomorphism. Show that if ϑ is a homomorphism then $\vartheta(1_G) = 1_H$. Show also that if g and h are elements of G with h being the inverse of g (with respect to the operation $\circ$), then $\vartheta(h)$ is the inverse of $\vartheta(g)$ (with respect to the operation $*$). Define the kernel and the image of ϑ, and state the homomorphism theorem.

Let G be the group of all 2×2 matrices of the form

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix},$$

with a_1, a_2, a_3 being real numbers and a_1, a_3 being non-zero. Define maps θ, ϕ by $\theta(A) = a_1$ and $\phi(A) = a_2$. Decide which of θ and ϕ are homomorphisms, calculating the kernel and image of the map(s) which are homomorphisms. Deduce that G has a (proper) normal subgroup N with abelian quotient group and also that N has a normal abelian subgroup K with N/K abelian.

5. State Lagrange's Theorem and use it to show that if a group G has an element of order k then k divides $|G|$. Let $A(4)$ denote the alternating group of the 12 even permutations on 4 letters. List the elements of $A(4)$ together with their orders. Give examples of the following justifying any assertions you make:

(1) A group G with a divisor d of $|G|$ (for $d \neq |G|$) but no elements of order d,

(2) A group G with a non-cyclic (proper) subgroup,

(3) A non-cyclic group in which every (proper) subgroup is cyclic.

6. Let G be a group. Define the terms *G-set*, *orbit* and *stabilizer*. State the orbit-stabilizer theorem.

Show that the set of subgroups of G is itself a G-set under conjugation (so $g \circ H = gHg^{-1}$) and give explicit descriptions of the orbit of a subgroup H of G and also of the stabilizer $N_G(H)$ of H in this case.

Determine the normalizer of H in G in the following cases:

(a) $G = D(4) = \langle x, y : x^4 = 1 = y^2, xy = yx^{-1} \rangle$ and $H = \langle x \rangle$,

(b) $G = D(4)$ and $H = \langle y \rangle$,

(c) $G = S(3)$ and $H = \{1, (1\ 2)\}$.

7. State the Sylow theorems and show that a group G has a unique Sylow p-subgroup if and only if the Sylow p-subgroups of G are normal.

Prove the following:

1. A group with 15 elements is cyclic,
2. A group with 56 elements either has a normal Sylow 2-subgroup or a normal Sylow 7-subgroup,
3. A group with 30 elements has an element of order 15.

8. State the Jordan-Hölder Theorem explaining the terms you use.

Let H, K be subgroups of a group with K a normal subgroup of H and H/K being of prime order. Prove that there is no normal subgroup L of H with $K < L < H$. Find composition series for each of the following, justifying an assertions you make:

(1) a cyclic group of order 4,

(2) a non-cyclic group of order 4,

(3) a cyclic group of order 15,

(4) the alternating group $A(4)$,

(5) the dihedral group with 20 elements.