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In this exam, the following notation and the following standard metrics are used. Whenever one of the sets listed below is mentioned, you may assume that it is equipped with its standard metric unless it is explicitly stated that a different metric is to be used.

a) The standard metric on $\mathbb{R}^n$ and on any subset of $\mathbb{R}^n$ is given by

$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

(where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are elements of $\mathbb{R}^n$.)

b) $\{0,1\}^{\mathbb{N}}$ denotes the set of all sequences $(s_n)_{n\geq 0}$ with each s_n equal either to 0 or to 1. The standard metric on $\{0,1\}^{\mathbb{N}}$ is given by

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ \frac{1}{2^n} & \text{if } n \text{ is smallest with } s_n \neq t_n \end{cases}$$

(where $s = (s_n)$ and $t = (t_n)$ are elements of $\{0,1\}^{\mathbb{N}}$).

c) For $a, b \in \mathbb{R}$ with $a < b$, $C[a,b]$ denotes the set of continuous functions $f : [a,b] \to \mathbb{R}$. The standard metric on $C[a,b]$ is the L^∞ metric d given by

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|.$$

Another metric on $C[a,b]$ which will be used (when explicitly stated) is the L^1 metric e given by

$$e(f,g) = \int_a^b |f(x) - g(x)| \, \mathrm{d}x.$$

d) If (X,d) and (Y,e) are metric spaces, then the standard metric D on the product $X \times Y$ is given by

$$D((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) + e(y_1,y_2).$$

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1. Let X be a set. What does it mean for a function $d : X \times X \to \mathbb{R}$ to be a *metric* on X? [4 marks]

(i) Let X be any set. Show that the function $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on X. [3 marks]

(ii) Let (X, d) and (Y, e) be any metric spaces (do *not* assume that d is the metric of (i)). Show that the function $D : (X \times Y) \times (X \times Y) \to \mathbb{R}$ defined by

$$D((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), e(y_1, y_2))$$

is a metric on $X \times Y$.

(You may assume without proof that $\max(a+b, c+d) \leq \max(a, c) + \max(b, d)$ for any real numbers a, b, c, and d.) [5 marks]

(iii) Show that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = (x - y)^2$ is not a metric on $\mathbb{R}$. [3 marks]

(iv) Let $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined as follows: given $m, n \in \mathbb{N}$, write each in base 10 in the normal way. If one has fewer digits than the other, then put 0s at the front of the shorter one to make them the same length. Then $d(m, n)$ is the number of positions at which m and n differ. (For example, $d(10325, 27) = d(10325, 00027) = 3$, since 10325 and 00027 differ in the 1st, 3rd, and 5th positions.)

Show that d is a metric on $\mathbb{N}$. [5 marks]

2. (i) Let $e : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be defined by

$$e(s,t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{2^n},$$

where $s = (s_n)$ and $t = (t_n)$. Show that e is a metric on $\{0,1\}^{\mathbb{N}}$. [4 marks]

(ii) Calculate $e((0110)^\infty, (1100)^\infty)$. Give an exact numerical answer. [3 marks]

($(0110)^\infty$ and $(1100)^\infty$ denote the elements $0110\,0110\,0110\ldots$ and $1100\,1100\,1100\ldots$ of $\{0,1\}^{\mathbb{N}}$.)

Use the standard metric d on $\{0,1\}^{\mathbb{N}}$ in the remainder of the question.

(iii) What does it mean for a sequence $(x_n)_{n\geq 0}$ in a metric space (X,d) to *converge* to a limit $\ell \in X$? [2 marks]

Show that the sequence $(s^{(n)})_{n\geq 0}$ in $\{0,1\}^{\mathbb{N}}$, where $s^{(n)} = (01)^n 0^\infty$, converges to $(01)^\infty$. [4 marks]

(Here $(01)^n 0^\infty$ denotes the element $01\,01\,01\ldots01\,0000000\ldots$ of $\{0,1\}^{\mathbb{N}}$, where the word 01 is repeated n times.)

(iv) Let (X,d) and (Y,e) be metric spaces. What does it mean for a function $f : X \to Y$ to be continuous? [2 marks]

Show that the function $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ defined by

$$f(s) = \sum_{n=0}^{\infty} \frac{s_n}{2^n}$$

is continuous. [5 marks]

3. (i) Let d and e denote the L^∞ and L^1 metrics on $X = C[0,1]$.

Calculate $d(f,g)$ and $e(f,g)$, where

$$\begin{aligned} f(x) &= x+1 \qquad \text{and} \\ g(x) &= x^2. \end{aligned}$$

(You should give exact numerical answers.) [4 marks]

(ii) Let (X,d) and (Y,e) be metric spaces. What does it mean for a function $f : X \to Y$ to be an *isometry*? What does it mean for f to be a *homeomorphism*? [4 marks]

(iii) Use the L^∞ metric d on $X = C[0,1]$ in this part of the question.

(a) Let A and B be the subsets of X defined by

$$\begin{aligned} A &= \{f \in C[0,1] \,:\, f(0) \geq 0\} \qquad \text{and} \\ B &= \{f \in C[0,1] \,:\, f(1) \geq 1\}. \end{aligned}$$

Show that the function $\mathcal{F} : A \to B$ defined by

$$\mathcal{F}(f)(x) = 1 + f(1-x)$$

is an isometry. [5 marks]

(b) Let C and D be the subsets of X defined by

$$\begin{aligned} C &= \{f \in C[0,1] \,:\, f(x) \in [0,1] \text{ for all } x \in [0,1]\} \\ D &= \{f \in C[0,1] \,:\, f(x) \in [0,2] \text{ for all } x \in [0,1]\}. \end{aligned}$$

Show that C and D are not isometric. [2 marks]

Show that C and D are homeomorphic. [5 marks]

4. Let (X,d) be a metric space, and A be a subset of X.

(i) What does it mean for A to be *open* in X? What does it mean for A to be *closed* in X? [4 marks]

(ii) In each of the following examples, state whether or not A is open in X, and whether or not A is closed in X. Justify your statements carefully using the definitions. (In each part, you should state explicitly whether or not A is open in X, and whether or not A is closed in X, and justify each of these two statements.)

(a) $X = \mathbb{R}$, $A = (0,1)$. [3 marks]

(b) $X = \mathbb{R}$, $A = \mathbb{R}$. [3 marks]

(c) $X = C[0,1]$, $A = \{f \in C[0,1] \,:\, f(1/2) = 0\}$. [5 marks]

(d) $X = \{0,1\}^{\mathbb{N}}$, $A = \{s \in \{0,1\}^{\mathbb{N}} \;:\; \text{the word 11 doesn't occur in } s\}$. [5 marks]

5. (i) Let $f_n : [0,1] \to \mathbb{R}$ for $n \geq 0$, and let $f : [0,1] \to \mathbb{R}$ (you should not assume that these functions are continuous).

(a) What does it mean for the sequence (f_n) to converge *pointwise* to f? What does it mean for it to converge *uniformly* to f? [3 marks]

(b) Give an example of a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ which converges pointwise to a function $f : [0,1] \to \mathbb{R}$ which is not continuous. (You should explain why it is true that (f_n) converges pointwise to f, but are not required to prove that these functions are continuous/discontinuous.) [3 marks]

(ii) Let $X = \{f \in C[0,1] : f(0) = 0,\ f(1) = 1\}$. You may assume that X is complete. Define a function $\mathcal{F} : X \to X$ by

$$\mathcal{F}(f)(x) = \begin{cases} \frac{f(3x)}{2} & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{1+f(3x-2)}{2} & \text{if } \frac{2}{3} \leq x \leq 1. \end{cases}$$

(a) Let $f \in X$ be given by $f(x) = x$. Give rough sketches of the graphs of f, $\mathcal{F}(f)$, and $\mathcal{F}(\mathcal{F}(f))$. [5 marks]

(b) Show that $\mathcal{F}$ is a contraction map, and explain why this means that there is some $g \in X$ such that, for any $f_0 \in X$, the sequence (f_n) defined inductively by $f_n = \mathcal{F}(f_{n-1})$ converges to g. [4 marks]

(c) Show that, for every $n \geq 1$, g is constant on the interval $[\frac{1}{3^n}, \frac{2}{3^n}]$.

[5 marks]

6. (i) Let (X, d) be a metric space.

(a) What does it mean for a sequence (x_n) in X to be *Cauchy*?

(b) What does it mean for (X, d) to be *complete*?

(c) What does it mean for a function $f : X \to X$ to be a *contraction map*? State the contraction mapping theorem. [8 marks]

(ii) Let $X = \{y \in C\,[1/2, 3/2] \,:\, y(x) \in [-1/2, 1/2] \text{ for all } x \in [1/2, 3/2]\}$. You may assume that X is complete.

Given $y \in X$, define $\mathcal{F}(y) \in C[1/2, 3/2]$ by

$$\mathcal{F}(y)(x) = \int_1^x \sin(z)e^{-zy(z)^2}\,\mathrm{d}z.$$

(a) Show that $\mathcal{F}(y)(x) \in [-1/2, 1/2]$ for all $x \in [1/2, 3/2]$, so that $\mathcal{F} : X \to X$. [4 marks]

(b) Show that, for all $y_1, y_2 \in X$, $\quad d(\mathcal{F}(y_1), \mathcal{F}(y_2)) \leq 3d(y_1, y_2)/4$.

[5 marks]

(c) Explain why this means that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x)e^{-xy^2}, \qquad y(1) = 0$$

has a unique solution $y(x)$ in X.

[3 marks]

7. (i) Let $k \geq 1$ be an integer, and let $X^{(k)}$ be the set of all non-empty compact subsets of $\mathbb{R}^k$.

Define the *Hausdorff distance* $\rho(A, B)$ between two elements of $X^{(k)}$. [3 marks]

In the remainder of the question you may assume without proof that ρ is a metric on $X^{(k)}$, and that $(X^{(k)}, \rho)$ is complete. You may also use any standard results from lectures, provided that you state them accurately.

(ii) Determine $\rho(A, B)$ for each of the following $A, B \in X^{(2)}$. Give brief explanations for your answers. (Recall that $\overline{B}_r((x_1, x_2))$ denotes the closed ball of radius r centred on $(x_1, x_2) \in \mathbb{R}^2$.)

(a) $A = \{(1, 1)\}$, $B = \{(4, 5)\}$. [2 marks]

(b) $A = \overline{B}_2((0, 0))$, $B = \{(1, 0)\}$. [2 marks]

(c) $A = \{(x, 0) \,:\, x \in [0, 2]\}$, $B = \{(1, y) \,:\, y \in [0, 2]\}$. [2 marks]

(iii) Define functions $f_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ for each pair of integers i, j with $0 \leq i \leq 2$, $0 \leq j \leq 2$, and i, j not both equal to 1, by

$$f_{i,j}(x, y) = \left(\frac{x+i}{3}, \frac{y+j}{3} \right).$$

Define $F : X^{(2)} \to X^{(2)}$ by

$$F(A) = f_{0,0}(A) \cup f_{0,1}(A) \cup f_{0,2}(A) \cup f_{1,0}(A) \cup f_{1,2}(A) \cup f_{2,0}(A) \cup f_{2,1}(A) \cup f_{2,2}(A).$$

(You may assume without proof that $F(A)$ is compact, and so is an element of $X^{(2)}$, for every $A \in X^{(2)}$.)

(a) Show that $\rho(F(A), F(B)) \leq \rho(A, B)/3$ for all $A, B \in X^{(2)}$. [4 marks]

(b) Let $A_0 = \{(x, y) \in \mathbb{R}^2 \,:\, 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$. Let A_n be defined inductively for $n \geq 1$ by $A_n = F(A_{n-1})$. Give a rough sketch of A_0, A_1, and A_2. [4 marks]

(c) Explain why the sequence (A_n) converges to some $A \in X^{(2)}$. [3 marks]

8. (i) What does it mean for a metric space (X, d) to be *sequentially compact*?

(a) Show that a subset A of a sequentially compact metric space (X, d) is itself sequentially compact if and only if it is closed in X. You may assume without proof that A is closed in X if and only if every convergent sequence in X all of whose terms lie in A has limit in A.

(b) Show that if (X, d) and (Y, e) are sequentially compact metric spaces, then so is their product $(X \times Y, D)$ (where D is the standard product metric). You may assume without proof that, if (x_n) and (y_n) are sequences in X and Y, then $(x_n, y_n) \to (x, y)$ in $X \times Y$ if and only if $x_n \to x$ and $y_n \to y$. [7 marks]

(ii) Determine whether or not each of the following metric spaces is sequentially compact. You may assume without proof that any closed interval $[a, b]$ is sequentially compact. You should give clear reasons for your answers, but are not required to give detailed proofs.

(a) $(0, 1]$. [2 marks]

(b) $[0, 1] \cup [2, 3]$. [2 marks]

(c) $[0, 1] \times (0, 1)$. [4 marks]

(d) $\{f \in C[0, 1] \, : \, f(x) \in [0, 1] \text{ for all } x \in [0, 1]\}$.

(Use the L^∞ metric on $C[0, 1]$.) [5 marks]