In this exam, the following notation and the following standard metrics are used. Whenever one of the sets listed below is mentioned, you may assume that it is equipped with its standard metric unless it is explicitly stated that a different metric is to be used.

a) The standard metric on $\mathbb{R}^n$ and on any subset of $\mathbb{R}^n$ is given by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

(where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are elements of $\mathbb{R}^n$.)

b) $\{0,1\}^{\mathbb{N}}$ denotes the set of all sequences $(s_n)_{n\geq 0}$ with each s_n equal either to 0 or to 1. The standard metric on $\{0,1\}^{\mathbb{N}}$ is given by

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ \frac{1}{2^n} & \text{if } n \text{ is smallest with } s_n \neq t_n \end{cases}$$

(where $s = (s_n)$ and $t = (t_n)$ are elements of $\{0,1\}^{\mathbb{N}}$).

c) For $a, b \in \mathbb{R}$ with $a < b$, $C[a,b]$ denotes the set of continuous functions $f : [a,b] \to \mathbb{R}$. The standard metric on $C[a,b]$ is the L^∞ metric d given by

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|.$$

Another metric on $C[a,b]$ which will be used (when explicitly stated) is the L^1 metric e given by

$$e(f,g) = \int_a^b |f(x) - g(x)| \, \mathrm{d}x.$$

d) If (X,d) and (Y,e) are metric spaces, then the standard metric D on the product $X \times Y$ is given by

$$D((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) + e(y_1,y_2).$$

1. Let X be a set. What does it mean for a function $d : X \times X \to \mathbb{R}$ to be a *metric* on X? [4 marks]

(i) Let X be any set. Show that the function $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on X. [3 marks]

(ii) Let d be any metric on a set X. Show that the function $e : X \times X \to \mathbb{R}$ defined by

$$e(x,y) = \min(d(x,y), 1)$$

is also a metric on X. [5 marks]

(iii) Let d be any metric on a non-empty set X. Show that the function $e : X \times X \to \mathbb{R}$ defined by

$$e(x,y) = \max(d(x,y), 1)$$

is *not* a metric on X. [3 marks]

(iv) Let X be the set of all closed intervals in $\mathbb{R}$, i.e.

$$X = \{[a,b] \,:\, a, b \in \mathbb{R},\ a < b\}.$$

Show that the function $d : X \times X \to \mathbb{R}$ defined by

$$d([a_1, b_1], [a_2, b_2]) = \max(|a_1 - a_2|, |b_1 - b_2|)$$

is a metric on X. (You may use the standard triangle inequality on $\mathbb{R}$ without proof.) [5 marks]

2. (i) What does it mean for two metrics d and e on a set X to be *equivalent*? [2 marks]

(ii) Show that two metrics d and e on a set X are equivalent if and only if for all $x \in X$ and all $\epsilon > 0$, there exists $\delta > 0$ such that

$$B^d_\delta(x) \subseteq B^e_\epsilon(x) \qquad \text{and} \qquad B^e_\delta(x) \subseteq B^d_\epsilon(x).$$

(Here $B^d_r(x)$ and $B^e_r(x)$ denote the open balls about x of radius r determined using the metrics d and e respectively. Recall that a subset U of X is d-open if and only if for every $x \in U$ there is some $\epsilon > 0$ with $B^d_\epsilon(x) \subseteq U$.) [4 marks]

(iii) In this part of the question, you may assume without proof that each d and each e is a metric.

(a) Let $X = \mathbb{R}^2$. Let d be the standard metric, and e be the metric defined by

$$e((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

By showing that $e(x, y) \leq d(x, y) \leq \sqrt{2}e(x, y)$ for all $x, y \in \mathbb{R}^2$, or otherwise, show that d and e are equivalent. [5 marks]

(b) Let $X = \mathbb{R}$. Let d be the standard metric, and e be the discrete metric defined by

$$e(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

By considering the ball $B^e_{1/2}(0)$, or otherwise, show that d and e are not equivalent. [4 marks]

(c) Let $X = C[0, 1]$. Show that the L^∞ metric d and the L^1 metric e on X are not equivalent. [5 marks]

(*Hint:* You may wish to consider $d(f, g_n)$ and $e(f, g_n)$, where $f(x) = 0$ is the zero function and $g_n(x) = x^n$ for each $n \geq 1$.)

3. (i) Let $e : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be defined by

$$e(s,t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{2^n},$$

where $s = (s_n)$ and $t = (t_n)$. Show that e is a metric on $\{0,1\}^{\mathbb{N}}$. [4 marks]

(ii) Calculate $e((001)^\infty, (010)^\infty)$. Give an exact numerical answer. [3 marks]
(Here $(001)^\infty$ and $(010)^\infty$ denote the elements $001\,001\,001\,\ldots$ and $010\,010\,010\,\ldots$ of $\{0,1\}^{\mathbb{N}}$.)

Use the standard metric d on $\{0,1\}^{\mathbb{N}}$ in the remainder of the question.

(iii) What does it mean for a sequence $(x_n)_{n\geq 0}$ in a metric space (X,d) to *converge* to a limit $\ell \in X$? [2 marks]

Show that the sequence $(s^{(n)})_{n\geq 0}$ in $\{0,1\}^{\mathbb{N}}$, where $s^{(n)} = (10)^n 1^\infty$, converges to $(10)^\infty$. [4 marks]
(Here $(10)^n 1^\infty$ denotes the element $10\,10\,10 \ldots 10\,1111111\ldots$ of $\{0,1\}^{\mathbb{N}}$, where the word 10 is repeated n times.)

(iv) What does it mean for two metric spaces (X,d) and (Y,e) to be *homeomorphic*? [2 marks]

Show that $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$. [5 marks]

(*Hint:* Consider a function $f : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ which interleaves the two input sequences. It will reduce the amount of work you need to do if you use a result which gives conditions under which a continuous bijection must be a homeomorphism – you may use this result without proof, but should state it accurately. You may assume without proof that $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is compact.)

4. Let (X, d) be a metric space, and A be a subset of X.

(i) Define the *boundary* ∂A of A in X. What does it mean for A to be *open* in X? What does it mean for A to be *closed* in X? [4 marks]

(ii) In each of the following examples, show (with careful arguments) that the boundary of A in X is as stated, and say whether or not A is open in X, and whether or not A is closed in X.

(a) $X = \mathbb{R}$, $A = [0, 1)$. Then $\partial A = \{0, 1\}$. [4 marks]

(b) $X = \mathbb{R}$, $A = \mathbb{Z}$. Then $\partial A = \mathbb{Z}$. [3 marks]

(c) $X = C[0, 1]$, $A = \{f \in C[0, 1] : f(0) < f(1)\}$.
Then $\partial A = \{f \in C[0, 1] : f(0) = f(1)\}$. [5 marks]

(d) $X = \{0, 1\}^{\mathbb{N}}$, $A = \{s \in \{0, 1\}^{\mathbb{N}} : s_n = 1$ for only finitely many $n\}$.
Then $\partial A = X$. [4 marks]

(*Hint:* the sequences in A are those of the form $s_0 s_1 \dots s_k 0^\infty$.)

5. (i) Let (X, d) be a metric space.

(a) What does it mean for a sequence (x_n) in X to be *Cauchy*?

(b) What does it mean for (X, d) to be *complete*?

(c) What does it mean for a function $f : X \to X$ to be a *contraction map*? State the contraction mapping theorem. [8 marks]

(ii) Let $X = \{y \in C[-1/2, 1/2] \,:\, y(x) \in [-1/2, 1/2] \text{ for all } x\}$. You may assume that X is complete.

Given $y \in X$, define $\mathcal{F}(y) \in C[-1/2, 1/2]$ by

$$\mathcal{F}(y)(x) = \int_0^x \cos(zy(z))\,\mathrm{d}z.$$

(a) Show that $\mathcal{F}(y)(x) \in [-1/2, 1/2]$ for all $x \in [-1/2, 1/2]$, so that $\mathcal{F} : X \to X$. [4 marks]

(b) Show that, for all $y_1, y_2 \in X$, $\quad d(\mathcal{F}(y_1), \mathcal{F}(y_2)) \leq d(y_1, y_2)/4$.

[5 marks]

(c) Explain why this means that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(xy), \qquad y(0) = 0$$

has a unique solution $y(x)$ in X. [3 marks]

6. (i) Let $k \geq 1$ be an integer, and let $X^{(k)}$ be the set of all non-empty compact subsets of $\mathbb{R}^k$.

Define the *Hausdorff distance* $\rho(A, B)$ between two elements of $X^{(k)}$. [3 marks]

In the remainder of the question you may assume without proof that ρ is a metric on $X^{(k)}$, and that $(X^{(k)}, \rho)$ is complete. You may also use any standard results from lectures, provided that you state them accurately.

(ii) Determine $\rho(A, B)$ for each of the following $A, B \in X^{(2)}$. Give brief explanations for your answers. (Recall that $\overline{B}_r((x_1, x_2))$ denotes the closed ball of radius r centred on $(x_1, x_2) \in \mathbb{R}^2$.)

(a) $A = \{(0, 0)\}$, $B = \{(2, 0)\}$. [2 marks]

(b) $A = \{(0, 0)\}$, $B = \overline{B}_1((2, 0))$. [2 marks]

(c) $A = \overline{B}_2((0, 0))$, $B = \{(x, 0) \,:\, x \in [-1, 1]\}$. [2 marks]

(iii) Let the functions $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f_1(x, y) = (x/2, y/2)$, $f_2(x, y) = (x/2, y/2) + (1/2, 0)$, and $f_3(x, y) = (x/2, y/2) + (1/4, \sqrt{3}/4)$.

Define $F : X^{(2)} \to X^{(2)}$ by $F(A) = f_1(A) \cup f_2(A) \cup f_3(A)$. (You may assume without proof that $F(A)$ is compact, and so is an element of $X^{(2)}$.)

(a) Show that $\rho(F(A), F(B)) \leq \rho(A, B)/2$ for all $A, B \in X^{(2)}$. [4 marks]

(b) Let A_0 be the equilateral triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1/2, \sqrt{3}/2)$. (So

$$A_0 = \{(x, y) \in \mathbb{R}^2 \,:\, 0 \leq y \leq \sqrt{3}/2, \;\; y/\sqrt{3} \leq x \leq 1 - y/\sqrt{3}\}.)$$

Let A_n be defined inductively for $n \geq 1$ by $A_n = F(A_{n-1})$. Give a rough sketch of A_0, A_1, and A_2. [4 marks]

(c) Explain why the sequence (A_n) converges to some $A \in X^{(2)}$. [3 marks]

7. (i) What does it mean for a metric space (X, d) to be *sequentially compact*? Show

(a) that a sequentially compact metric space is bounded;

(b) that a subset A of a sequentially compact metric space (X, d) is itself sequentially compact if and only if it is closed in X. (You may assume without proof that A is closed in X if and only if every convergent sequence in X all of whose terms lie in A has limit in A.) [7 marks]

(ii) Determine (giving reasons) whether or not each of the following metric spaces is sequentially compact. You may assume without proof that any closed interval $[a, b]$ is sequentially compact.

(a) $[0, 1)$. [2 marks]

(b) $\{0, 1\}$. [2 marks]

(c) $C[0, 1]$. [4 marks]

(d) $\{0, 1\}^{\mathbb{N}}$. [5 marks]

8. (i) What does it mean for a metric space (X, d) to be *compact*? Explain any terms which you use in the definition. [4 marks]

(ii) Let (X, d) be a compact metric space. Let $\mathcal{P}$ be a property of subsets of X: that is, for each subset A of X, $\mathcal{P}(A)$ is a statement which is either true or false. Suppose that

- For every $x \in X$, there is an open set U_x containing x such that $\mathcal{P}(U_x)$ is true, and
- If A and B are subsets of X such that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are true, then $\mathcal{P}(A \cup B)$ is also true.

Show that $\mathcal{P}(X)$ is true. [5 marks]

(iii) Show that if (X, d) is compact, then any continuous function $f : X \to \mathbb{R}$ is bounded. [6 marks]

(iv) Let X be a compact subset of $\mathbb{R}^2$, with the property that any two points of X can be joined by a path in X consisting of a finite number of straight line segments. Suppose that for each $x \in X$, there is an open subset U_x of X containing x and a number N_x such that any two points of U_x can be joined by a path in X consisting of fewer than N_x straight line segments.

Show that there is a number N such that any two points of X can be joined by a path in X consisting of fewer than N straight line segments. [5 marks]