(January 2006)

Time allowed: Two Hours and a Half.

You may attempt as many problems as you like. The best FIVE answers will be taken into account. Each question carries the same weight.

1.

(i) Find all zeros, poles and their orders of the meromorphic function

$$f(z) = \frac{(z^2+4)^2\,(z-2)^2}{(z^2-1)^2}$$

on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. [7 marks]

(ii) Using the coordinate $\tau = 1/z$ at ∞, find the principal part of the Laurent expansion of $f(z)$ at ∞. [8 marks]

(iii) Find the radii of convergence of the Laurent expansion (ii) of $f(z)$ at ∞. [5 marks]

2.

(i) Find the residues of the meromorphic differential

$$\omega = \frac{2z+1}{z^2-z}\,dz$$

at its poles on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. [14 marks]

(ii) Write down the sum of the residues. Is this predicted by the residue theorem? [2 marks]

(iii) Formulate the residue theorem for a compact Riemann surface. Sketch a proof of the theorem. [4 marks]

3.

Consider the holomorphic map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by the meromorphic function

$$f(z) = \frac{z^2-2z}{z^2-2z+1}, \quad z \in \mathbb{C}.$$

(i) Find all ramification points and their orders for f. [12 marks]

(ii) Find the degree of f. [4 marks]

(iii) Check the statement of the Riemann-Hurwitz Formula for f. [4 marks]

4.

(i) Find the Riemann surface of the multi-valued holomorphic function

$$w = \sqrt{z-4} - \sqrt{z+6}, \quad z \in \mathbb{C}.$$

[15 marks]

(ii) Find the corresponding algebraic Riemann surface. [5 marks]

5.

Consider the elliptic curve X given by the equation

$$\eta_0\eta_2^2 = \eta_1^3 + 2\eta_0^2\eta_1$$

in projective coordinates $(\eta_0 : \eta_1 : \eta_2)$, or (equivalently) by the equation $y^2 = x^3 + 2x$ in the affine coordinates $x = \eta_1/\eta_0$, $y = \eta_2/\eta_0$.

(i) Show that the meromorphic differential

$$\omega = \frac{dx}{y}$$

is holomorphic on X and has no zeros. [15 marks]

(ii) Give evidence why X is not isomorphic to the Riemann sphere $\overline{\mathbb{C}}$. [2 marks]

Show that X is a torus. [3 marks]

6.

(i) Formulate the Riemann-Roch Theorem for a compact Riemann surface of genus g. [5 marks]

Explain clearly what is meant by each term in the theorem. [5 marks]

(ii) Assume that X is a compact Riemann surface of genus 5 and A_1, A_2 are two distinct points of X.

Using the Riemann-Roch Theorem, find the dimension of the space of meromorphic functions on X which have poles of the orders not more than 5 at A_1, A_2 and which are holomorphic in all other points of X. [4 marks]

Prove that there exists a meromorphic function f on X which at the points A_1, A_2 has poles of orders exactly 5, and f is holomorphic in all other points of X. [6 marks]

7.

Using a triangulation or otherwise, find the Euler characteristic of

(i) the closed plane domain shown shaded on the picture below;
[10 marks]

(ii) the torus (you can obtain the torus by gluing together pairs of opposite edges of the rectangle in the same direction, see diagram below). [10 marks]

8.

(i) Prove that the Riemann surfaces $R_1 = \overline{\mathbb{C}} - \{1, 2, 3\}$ and $R_2 = \overline{\mathbb{C}} - \{\infty, 0, 1\}$ are isomorphic. [8 marks]

(ii) Prove that the Riemann surfaces $R_1 = \overline{\mathbb{C}} - \{1, 2, 3, 4\}$ and $R_2 = \overline{\mathbb{C}} - \{-1, 1, 0, 5\}$ are not isomorphic. [12 marks]

9.

(i) What are canonical domains? [2 marks]

(ii) Explain their similarity and their difference. [5 marks]

(iii) Formulate the Riemann uniformisation theorem for Riemann surfaces. [4 marks]

(iv) Explain clearly what is meant by each term in the theorem. [5 marks]

(v) Give an example of the uniformisation of a Riemann surface which is different from the canonical domains. [4 marks]