THE UNIVERSITY
of LIVERPOOL

Math 329 — January 2003 Exam

"Numerical Linear Algebra and its Applications"

Full marks will be awarded for complete answers to SIX questions. Only the best 6 answers will be taken into account. Note that each question carries a total of 15 marks that are distributed as stated.

1. **[15 marks]**

(1) Using the matrix p-norm definition, prove that

$$\|A_1 + 3A_2 - 2A_3\|_p \le \|A_1\|_p + 3\|A_2\|_p + 2\|A_3\|_p$$

where $A_1,\ A_2,\ A_3 \in R^{n\times n}$. [5 marks]

(2) Construct a non-trivial 3×3 matrix A (i.e. $a_{ij} \ne 0$ for all i, j) such that [5 marks]

$$\|A\|_1 = 10, \quad \|A\|_\infty = 12.$$

(3) Given the eigenvalues of $A^T A - 7I$ as $\lambda(A^T A - 7I) = [2,\ \ -3,\ \ -2,\ \ 0]$, find the minimum singular value $\sigma_{min}(A)$ and 2-norm $\|A\|_2$ for some 4×4 matrix A. [5 marks]

2. **[15 marks]**

(1) Let $A \in R^{n\times n}$. To find $\lambda(A)$, the QR method implements (from $A_0 = A$ and for $k = 1, 2, 3, \ldots$) [7 marks]

$$\begin{cases} A_{k-1} = Q_k R_k, \\ A_k = R_k Q_k. \end{cases}$$

After n steps of the QR method, work out the iteration matrix P_n such that $A_n = P_n^T A_0 P_n$.

(2) For a matrix $A_{3\times 3}$, assume that there exists an orthogonal matrix P such that

$$P^\top A P = D = \begin{bmatrix} 10 & & \\ & 2 & \\ & & 3 \end{bmatrix}.$$

Use the perturbation theorem to predict the locations of eigenvalues $\mu_j(B)$ of the following matrix (keep <u>2 decimal digits</u> in answers) [8 marks]

$$B = A + \delta A \qquad \text{and} \qquad \delta A = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0.1 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

Hint. $\lambda(\delta A) = [0.1\ \ 0.2\ \ 0.3]$, $\qquad \lambda(\delta A^T \delta A) = [0.0074\ \ 0.0455\ \ 0.1071]$.

3. **[15 marks]**

(1) Explain how step k of the Gran-Schmidt method is proceeded. [5 marks]
Hint: You may assume that the kth equation is

$$a_k = q_1 u_{1k} + q_2 u_{2k} + \cdots + q_k u_{kk}.$$

(2) For the 3×3 unsymmetric matrix, [10 marks]

$$A = \begin{pmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{pmatrix},$$

complete the last step of a QR factorization by the Gram-Schmidt method (write down Q and R) given that $u_{11} = 557.9418,\ u_{12} = 187.0321, u_{22} = 0.0741,$

$$q_1 = \begin{pmatrix} -0.2671 \\ 0.9625 \\ -0.0484 \end{pmatrix}, q_2 = \begin{pmatrix} -0.7088 \\ -0.1621 \\ 0.6865 \end{pmatrix}.$$

(Keep 4 decimal digits in answers).

4. **[15 marks]**

(1) Let $N = 2^t$ for some positive integer t. Ignoring the full details of the permutation phase, what is the Cooley-Tukey Fast Fourier Transform (FFT) factoriasation for the Discrete Fourier Transform matrix A_N? Further explain why $A_N x$, for $x \in C^N$, takes only $O(N \log N)$ operations by the FFT. [5 marks]

(2) Implement the permutation phase of the FFT by shuffling the following vector [5 marks]

$$x = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 99 \\ 21 \\ 33 \\ 12 \\ 18 \end{pmatrix}.$$

(3) Given the first row r and first column c of a Toeplitz matrix [5 marks]

$$r = \begin{bmatrix} 2 & 3 & 5 & 5 \end{bmatrix}, \qquad c = \begin{bmatrix} 2 & 8 & 3 & 7 \end{bmatrix}^T,$$

what is the matrix T? Construct a suitable circulant matrix that embeds T.

5. **[15 marks]**

In the general n-dimensional case

(1) Write down the Givens matrix $P_{1,3}$ and verify that it is orthogonal. [5 marks]

(2) Show that the Householder matrix $P = P(v) = I - \frac{2}{v^T v} v v^T$ is both symmetric and orthogonal where $v \in R^n$ is nonzero. [5 marks]

(3) Let $x \in R^n$ admit the partition

$$x = \begin{pmatrix} \widetilde{x} \\ \overline{x} \end{pmatrix}$$

where $\widetilde{x} \in R^\ell$ and $\overline{x} \in R^{n-\ell}$ for some integer $0 \le \ell \le n-1$. In each of the following two cases, `without` forming $P(v)$, find a suitable vector $v \in R^n$ and the scalar α such that

$$P(v)x = \begin{pmatrix} \widetilde{x} \\ -\alpha \\ \mathbf{0} \end{pmatrix}$$

(3a) $n = 5$, $\ell = 2$ and $x = [\ 2009 \quad 2003 \quad 4 \quad -2 \quad -\sqrt{5}\ ]^T$. [2 marks]

(3b) $n = 4$, $\ell = 1$ and $x = [2004 \quad -2 \quad \sqrt{3} \quad 3\sqrt{2}]^T$. [3 marks]

Use <u>`exact`</u> arithmetic.

6. **[15 marks]**

(1) Find the companion matrix B for polynomial $P_4(x) = x^4 + 4x^3 - 2x^2 - 1$. [3 marks]

(2) For the above matrix B, work out the graph $G(B^T B)$ and then compute one singular value. Use <u>`exact`</u> arithmetic. [12 marks]

7. [**15 marks**]

(1) Define the functional $\Phi(x) = \frac{1}{2}x^T A^T A x - x^T A^T b$ for vectors $x,\ b \in R^n$ and matrix $A \in R^{n\times n}$. Prove that [5 marks]

$$\Phi(x) = \frac{1}{2}\|Ax - b\|_2^2 - \frac{1}{2}b^T b.$$

Further show that if A is orthogonal, the solution to the minimisation problem

$$\min_x \Phi(x)$$

is $x = A^T b$. [5 marks]
Hint. You may assume that for any $x,\ c \in R^n$ and matrix $B \in R^{n\times n}$,

$$\nabla(x^T B x) = 2Bx, \quad \nabla(x^T c) = c.$$

(2) Based on a symmetric positive definite matrix A, what is meant by vectors $p,\ q$ being A-conjugate? At the start of step k of a conjugate gradient method, given that the present residual r_{k-1} and the previous search direction p_{k-1} are respectively

$$r_{k-1} = \begin{pmatrix} -24.2182 \\ -10.3568 \\ -0.2858 \\ 8.3649 \end{pmatrix}, p_{k-1} = \begin{pmatrix} 28.1667 \\ 77.0500 \\ 128.6000 \\ 181.3405 \end{pmatrix}, \quad \text{and} \quad \frac{(p_{k-1}^T A r_{k-1})}{(p_{k-1}^T A p_{k-1})} = 0.0136,$$

construct the new search direction p_k that is A-conjugate to p_{k-1}.
(Keep 4 decimal digits in answers) [5 marks]

8. **[15 marks]**

Consider the numerical solution of the linear differential equation on the unit square Ω centred at $(1/2, 1/2)$, $\quad 4\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial y} + 16u = \cos(xy), \quad$ with $u = 1$ at the boundary Γ. Using the finite difference method (i.e. the usual central differences) with 9 internal points (equally spaced), ordered lexicographically as in Fig.1,

(1) Verify that the (node 9) equation at point $(x_3, y_3) = (3/4, 3/4)$ is [5 marks]

$$9u_6 + 32u_8 - 72u_9 = \frac{1}{2}\cos(\frac{9}{16}) - 39.$$

(2) Verify that the (node 5) equation at point $(x_2, y_2) = (1/2, 1/2)$ is [5 marks]

$$9u_2 + 32u_4 - 72u_5 + 32u_6 + 7u_8 = \frac{1}{2}\cos(\frac{1}{4}).$$

(3) Find a spiral ordering $r = [r_1,\ r_2,\ \cdots,\ r_9]$ in terms of the old orderings r_j (do not work out any matrix!). [5 marks]

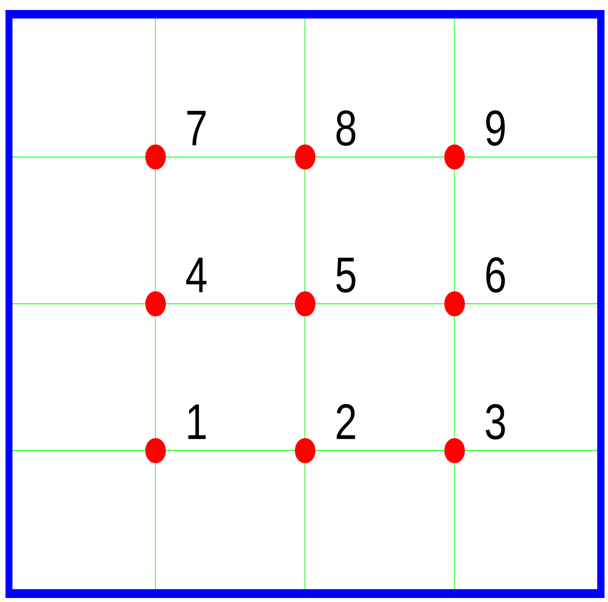

Figure 1: Lexicographical ordering of a FDM mesh