Math 329 — Jan 2000 Exam

"Numerical Linear Algebra and its Applications"

Full marks are given to the best answers of FIVE questions.
Each question carries a total of 20 marks that are distributed as stated.

I (20 marks)

1. For an $n \times n$ real matrix A, a definition of its eigenvalues $\lambda_i(A)$ is given by $Ax_i = \lambda_i x_i$, where x_i is a nonzero vector. Using this definition and norm definitions, prove the following for any index i

   (a) $|\lambda_i| \leq \|A\|_p$ for any $p-$norm; [5 marks]

   (b) $\lambda_i(A^{-1}) = 1/\lambda_i(A)$ if $\lambda_i(A) \neq 0$. [5 marks]

2. Given that for some $A_{3\times3}$, $\lambda_i(A^TA) = [0.9,\ 23.6,\ 25]$:

   (a) compute $\|A\|_2$; [5 marks]

   (b) find the spectral radius $\rho(A^{-1}A^{-T})$ where $A^{-T} = (A^T)^{-1}$. [5 marks]

II (20 marks)

Given the symmetric matrix

$$A = \begin{pmatrix} -15 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 2 & -2 \\ 0 & 0 & -2 & 14 \end{pmatrix},$$

1. Use the Householder transforms to tridiagonalise A i.e. find an orthogonal matrix P such that $P^TAP = T$ is tridiagonal.
   Hint. Take $sign(0) = +1$. [10 marks]

2. Use the Sturm sequence method or otherwise to isolate the eigenvalues of A. [10 marks]

III (20 marks)

Given the symmetric matrix A and the vector b:

$$A = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 7 \end{pmatrix}, \qquad \text{and} \qquad b = \begin{pmatrix} 2 \\ -5 \\ 19 \end{pmatrix},$$

find an approximate solution to the system $Ax = b$ by minimising the functional

$$\Phi(x) = \frac{1}{2} x^T A x - x^T b$$

along two successive search directions

$$p_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad p_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Use $x^{(0)} = [1\ 1\ 1]^T$.

IV (20 marks)

Let x_1, x_2, x_3, x_4 be four independent eigenvectors of a symmetric matrix $A_{4\times 4}$ (corresponding to four distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$). Take $q_1 = \overline{q}_1 / \|\overline{q}_1\|_2$ (with $\overline{q}_1 = \sum_{j=1}^{4} x_j$) as a known vector.

1. Derive the formulae of the Lanczos method for this matrix A with the starting vector q_1; [10 marks]

2. Determine the dimension of the Krylov space [10 marks]

$$\kappa(q_1, A, 4) = span(q_1, Aq_1, A^2 q_1, A^3 q_1).$$

Hint. The following Vandermonde matrix is non-singular:

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{pmatrix}.$$

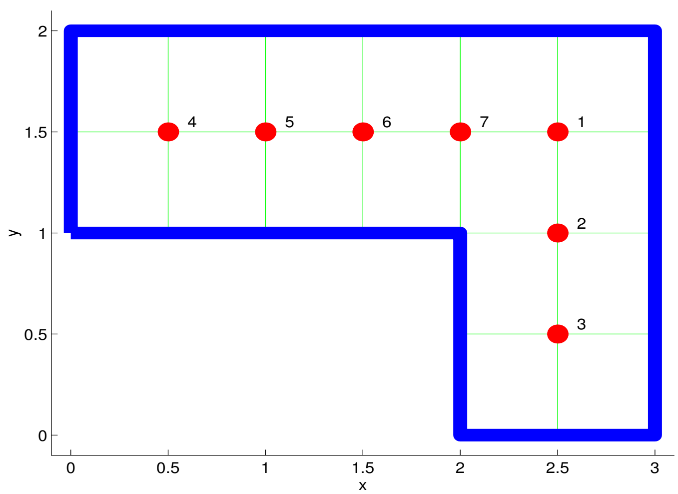
V (20 marks)

Consider a finite difference solution of the following differential equation for $u = u(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 y + 1,$$

defined in an L-shaped domain (see the boundary Γ in Figure 1) with zero boundary conditions (i.e. $u|_\Gamma = 0$); there will be 7 unknowns with the uniform mesh as set.

Figure 1: An L-shaped domain



1. Find a new node ordering suggested by the Cuthill-Mckee algorithm; [10 marks]

2. Write down the 7×7 linear system of equations for the new ordering. [10 marks]

VI (20 marks)

Given the cubic polynomial $f_3(x) = x^3 - 2x^2 - 400x + 800$,

1. Find its companion matrix A such that $det(xI - A) = f_3(x)$; [7 marks]

2. Use 1 step of the QR method (no shift) to obtain the approximate roots of $f_3(x)$. (Keep at least 3 significant figures.) [13 marks]

VII (20 marks)

1. Solve the following polynomial equation for λ [10 marks]

$$f_4(\lambda) = \begin{vmatrix} w_1 & w_2 & (\lambda - 2) & -1 & w_3 \\ (\lambda + 1) & 46 & 0 & 0 & 0 \\ 45 & 10 & 0 & 0 & (\lambda - 4) \\ 0 & (\lambda - 3) & 0 & 0 & 0 \\ 10 & 40 & 2 & (\lambda - 5) & w_4 \end{vmatrix} = 0,$$

where w_j's are some (real) scalars.

2. Find the companion matrix B for the above $f_4(\lambda)$; [6 marks]

3. If there exist some known and non-singular matrices L_j, U_j such that

$$L_5 L_4 L_3 L_2 L_1 B U_1 U_2 U_3 U_4 U_5 = I,$$

find the inverse matrix B^{-1} in terms of L_j, U_j. [4 marks]