MATH328 June 2007

In this paper bold-face quantities like $\mathbf{x}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Consider the infinitesimal line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2.$$

(a)

Write the metric $g_{\mu\nu}$ and its inverse in an explicit in matrix form.

[5 marks]

(b) Find the set of independent transformations of the form

$$t \to t + \epsilon A(t, x)$$
$$x \to x + \epsilon B(t, x) \ ,$$

where ϵ is an infinitesimal constant and the functions A and B have to be determined by the requirement that ds^2 is invariant. State what each transformation represents in space time.

[15 marks]

2. Consider the Poincare group in 1+2 dimensions, with the infinitesimal line element

$$ds^2 = dt^2 - dx^2 - dy^2.$$

(a) Write in matrix form the metric for this line element and its inverse. Write all the transformations under which this line element is invariant. Write down the generators associated with each transformation.

[5 marks]

(b) The Pauli-Lubanski vector in four dimensions is given by

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho} J_{\nu\sigma} P_\rho,$$

where $K_i = J_{i0}$ and $J_i = 1/2\epsilon_{ijk} J_{jk}$ are the generators of boosts and rotations, respectively, and P_ρ is the momentum four vector. Write down the four components of the Pauli-Lubanski vector for massless and massive particle states, in the case where the line element $ds^2 = dt^2 - dx^2 - dy^2$ is viewed as embedded in four space–time dimensions.

[15 marks]

3. The potential function of a two dimensional harmonic potential is

$$V(x,y) = \frac{1}{2}k(x^2+y^2).$$

(a) Write down the Lagrangian. [3 marks]

(b) Write down the Euler-Lagrange eqs. of motion. [3 marks]

(c) Write down the Hamiltonian. [5 marks]

(d) Write down the Lagrangian and Hamiltonian in polar coordinates (r,ϕ) with $(x = r\cos\phi\ ,\ y = r\sin\phi)$. [5 marks]

(e) How many constants of the motion are there? What are they?

[4 marks]

4. The annihilation and creation operators in real scalar field theory for a particle of momentum $\mathbf{p}$, $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ respectively, satisfy

$$\begin{aligned}[a(\mathbf{p}), a^\dagger(\mathbf{p}')] =& 2p^0\delta(\mathbf{p}-\mathbf{p}')(2\pi)^3, \\ [a(\mathbf{p}), a(\mathbf{p}')] =& 0, \\ [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] =& 0,\end{aligned}$$

where $p^0 = \sqrt{\mathbf{p}^2+m^2}$. n-particle states are defined by

$$|\mathbf{p}_1, \mathbf{p}_2, \dots \mathbf{p}_n >= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)\dots a^\dagger(\mathbf{p}_n)|0>,$$

where $|0>$ is the vacuum state satisfying $a(\mathbf{p})|0>=0$ for all $\mathbf{p}$.

(a) Show that

$$\begin{aligned}&< \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2 > \\ =&(2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2') + \delta(\mathbf{p}_1-\mathbf{p}_2')\delta(\mathbf{p}_2-\mathbf{p}_1')\}.\end{aligned}$$

[9 marks]

(b) Show that if we define the number operator

$$N = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a^\dagger(\mathbf{p})a(\mathbf{p}),$$

then it satisfies

$$[N, a^\dagger(\mathbf{p})] = a^\dagger(\mathbf{p})$$

and hence

$$N|\mathbf{p}_1 \dots \mathbf{p}_n >= n|\mathbf{p}_1 \dots \mathbf{p}_n >$$

[11 marks]

5. (a) Derive the conservation equation $\partial_\mu J_V^\mu = 0$ for the four vector current density $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

[10 marks]

(b) Show that the axial 4-vector current density $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi \ .$$

[10 marks]

6. Consider the simple unitary group $SU(3)$.

(a.) How many diagonal generators of the Lie algebra are there? Write down a representation of the diagonal generators in terms of 3×3 hermitian matrices. [3 marks]

(b.) What is the dimension of the group? Write down a representation of all generators in terms of 3×3 hermitian matrices. [3 marks]

(c.) What is the fundamental representation of $SU(3)$? Write down its decomposition in terms of a maximal subgroup.

[3 marks]

(d.) Draw the graphic illustration of the fundamental representation, indicating clearly the eigenvalues of each state under the diagonal generators.

[3 marks]

(e.) Find the product and the decomposition under the maximal subgroup of the fundamental times the anti-fundamental representations of $SU(3)$.

[4 marks]

(f.) Find the product and the decomposition under the maximal subgroup of the fundamental times the fundamental representations of $SU(3)$.

[4 marks]

7. The Lagrangian density for an interacting complex scalar field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2),$$

is

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2$$

with $\mu^2 < 0$ and $\lambda > 0$.

(a) What are the transformations under which $\mathcal{L}$ is invariant?

[2 marks]

(b) Show that it describes a massive field of mass $\sqrt{-2\mu^2}$ and one massless Goldstone boson.

[18 marks]