MATH328 June 2006

In this paper bold-face quantities like $\mathbf{x}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. (a) Consider the infinitesimal line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2.$$

Write the metric $g^{\mu\nu}$ in an explicit matrix form. Write $g_{\mu\nu}$ in matrix form. Find the set of independent transformations of the form

$$t \rightarrow t + \epsilon A(t, x)$$
$$x \rightarrow x + \epsilon B(t, x) \ ,$$

where ϵ is an infinitesimal constant and the functions A and B have to be determined, that leave ds^2 invariant. State what each transformation represents in space time. [10 marks]

(b) Consider the Poincare group in 1+2 dimensions, with the infinitesimal line element

$$ds^2 = dt^2 - dx^2 - dy^2.$$

The Pauli-Lubanski vector in four dimensions is given by

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\sigma\rho} J_{\nu\sigma} P_\rho,$$

where $K_i = J_{i0}$ and $J_i = 1/2\epsilon_{ijk} J_{ij}$ are the generators of boosts and rotations, respectively, and P_ρ is the momentum four vector. Write down the four components of the Pauli-Lubanski vector in the case of the 3 dimensional line element given above, for massless and massive particle states. [10 marks]

2. Suppose that we live on a two dimensional surface with a line element on it given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

(a convenient notation is $\theta^\mu \equiv (\theta, \phi)$ $(\mu = 1, 2)$)

Write the metric $g^{\mu\nu}$ in an explicit matrix form. Write $g_{\mu\nu}$ in matrix form.

Find the set of infinitesimal transformations of the form

$$\theta^\mu \rightarrow \theta^\mu + \epsilon \zeta^\mu(\theta, \phi)$$

for which the line element ds^2 is invariant. [20 marks]

3. (a) Let $\vec{J}$ and $\vec{K}$ be the generators of rotations and boosts, respectively. Show that

$$\vec{J}^2 - \vec{K}^2 \quad \text{and} \quad \vec{J} \cdot \vec{K}$$

are Lorentz invariants (*i.e.* that they commute with all the generators of the Lorentz group). [10 marks]

(b) Assume a representation (j_1, j_2) of $SU(2) \times SU(2)^\dagger$. How many states are there in this representation? How does this representation decompose in irreducible representations of $SU(2)_J$, where J is the total angular momentum? [10 marks]

4. Write down the Lagrangian for a particle of mass m in a potential $V(r, \phi)$ when referred to planar polar coordinates (r, ϕ). Hence show that the equations of motion are

$$m\ddot{r} - mr\dot{\phi}^2 = -\frac{\partial V}{\partial r},$$
$$mr\ddot{\phi} + 2m\dot{r}\dot{\phi} = -\frac{1}{r}\frac{\partial V}{\partial \phi}.$$

Derive the principle of conservation of angular momentum in the plane for an axisymmetric potential, and obtain the usual formula v^2/r for centripetal acceleration ($\dot{r} = 0$ and v is the particle's speed). [20 marks]

5. (a) Show that if the Hamiltonian is independent of a generalized coordinate q_0, then the conjugate momentum p_0 is a constant of the motion. Such coordinates are called **cyclic coordinates**. Give two examples of a physical system that has a cyclic coordinate. [5 marks]

(b) Show that in 3 dimensional spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\vec{x})$ is

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}) + V(\vec{x}).$$

Show that $p_\phi =$ constant when $\partial V/\partial\phi \equiv 0$ and interpret this result physically. [15 marks]

6. (i) The Lagrangian density for a real scalar field is given by

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

Show that the Hamiltonian H_0 is given by

$$H_0 = \tfrac{1}{2}\int[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2]d^3x.$$

[5 marks]

(ii) Show that if we take the usual canonical commutation relations,

$$\begin{aligned}[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] =& i\hbar\delta(\mathbf{x}-\mathbf{x}'),\\ [\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] =& 0,\\ [\pi(\mathbf{x},t),\pi(\mathbf{x}',t)] =& 0,\end{aligned}$$

the equations of motion are obtained from

$$i\hbar\dot{\phi} = [\phi, H_0] \text{ and } i\hbar\dot{\pi} = [\pi, H_0] \ .$$

[15 marks]

7. (a) Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$\begin{aligned}(\gamma^5)^2 =& 1\\ \gamma^{5\dagger} =& \gamma^5.\end{aligned}$$

[12 marks]

(b) Write each of $\gamma^1\gamma^2\gamma^3$ and $\gamma^0\gamma^2\gamma^3$ as a product $c\gamma^5\gamma^\nu$ for some $\nu = 0, 1, 2, 3$ and some number c. [5 marks]

(c) Show that

$$\text{trace}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}.$$

[You may assume $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.] [3 marks]

8. (a) Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Show that Maxwell's equation in four vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{e.m.} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}.$$

Show that $L_{e.m.}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu\Lambda$, where Λ is a scalar function. [10 marks]

(b) Show that imposing a local $U(1)$ symmetry forbids the photon from attaining a mass. [10 marks]