MATH327 - Stochastic Processes and Statistical Mechanics
January 2002

Time allowed: Two hours and a half

Full marks can be obtained for complete answers to FIVE questions.
Only the best FIVE answers will be counted.

1. Define the term Markovian, with respect to a discrete-time stochastic process.

A discrete-time Markov process has three states denoted 1, 2 and 3, and the transitions between them have probabilities

$$P(1 \to 2) = \alpha, \quad P(2 \to 1) = 0,$$
$$P(2 \to 3) = \beta, \quad P(3 \to 2) = \beta,$$
$$P(3 \to 1) = 0, \quad P(1 \to 3) = \alpha,$$

where $0 < \alpha, \beta < 1$ and $\alpha \neq \beta$. Consider the resulting Markov chain and obtain the stochastic matrix Q.

Find the eigenvalues and *right* eigenvectors of Q and obtain the probability distribution as the number of steps $n \to \infty$.

Initially the system is either in state 1 or state 2 with equal probability. Write down a vector describing the initial probability of the three-state system. Expand this vector with respect to the three right eigenvectors as a basis.

If $\alpha = \frac{1}{4}$, $\beta = \frac{3}{8}$, show that the probability to be in state 3 after n steps is

$$P(3, n) = \frac{1}{2} - \frac{1}{4}\left(\frac{1}{2^n} + \frac{1}{4^n}\right).$$

[20 marks]

2. The continuous-time Markov process $X(t)$ has three possible values x_1, x_2 and x_3. The transition rates are

$$W(x_1 \to x_2) = W(x_2 \to x_3) = 1$$
$$W(x_3 \to x_2) = W(x_2 \to x_1) = 2$$
$$W(x_3 \to x_1) = W(x_1 \to x_3) = 0.$$

Express the Master Equation in the matrix form

$$\frac{\partial}{\partial t}\mathbf{P}(t) = M\mathbf{P}(t),$$

where $\mathbf{P}(t) = \begin{pmatrix} P(x_1, t) \\ P(x_2, t) \\ P(x_3, t) \end{pmatrix}$ and M is a 3×3 matrix which you should specify.

Show that the eigenvalues of M are 0 and $-3 \pm \sqrt{2}$, and moreover show that the (suitably normalised) right eigenvector corresponding to the eigenvalue 0 is

$$\mathbf{X_0} = \frac{1}{7}\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.$$

(You do **not** need to compute the other eigenvectors.)

Now solve the Master Equation in the matrix form derived above, and so deduce that as $t \to \infty$, $\mathbf{P}(t) \to \mathbf{X_0}$, regardless of initial conditions. [20 marks]

3. For a stochastic process, define the probability transition rate $W(x \to x')$, $x \neq x'$ where x and x' are possible states of the process.

The possible states of a Markov process $X(t)$ are the positive integers. The transition rates are

$$W(x \to x') = \begin{cases} a & x' = x + 1, \\ bx & x' = x - 1, \\ 0 & \text{otherwise}, \end{cases}$$

where a and b are positive constants.

Write down the Master Equation for this process and hence show that

$$\overline{X(t)} - e^{-bt}\overline{X(0)} = \frac{a}{b}\left(1 - e^{-bt}\right).$$

Also find an expression for $\frac{d\overline{X^2(t)}}{dt}$ in terms of $\overline{X^2(t)}$ and $\overline{X(t)}$.

Hence, given $X(0) = 0$, find an expression for $\overline{X^2(t)}$. [20 marks]

4. The stochastic variable X satisfies the stochastic differential equation

$$\frac{dX}{dt} = -\frac{X}{t} + A(t),$$

where the stochastic quantity A satisfies

$$\overline{A(t)} = \alpha, \quad \text{and} \quad \overline{A(t)A(t')} = \alpha^2 + \alpha\delta(t - t'),$$

where α is a positive constant. Use an integrating factor technique or other means to show that the solution for $X(t)$ satisfying $X(1) = 0$ is such that

$$\overline{X(t)} = \frac{\alpha}{2t}(t^2 - 1),$$

and find a corresponding expression for

$$\sigma^2(t) = \overline{X^2(t)} - \left(\overline{X(t)}\right)^2.$$

Verify that, for large t,

$$\frac{\overline{X(t)}}{\sigma^2(t)} \approx \frac{3}{2}.$$

Give one example of a simple physical system which can be described by a stochastic differential equation. You should write down and describe the relevant equation, but need not solve it. [20 marks]

5. A system consists of a large number N of distinguishable, weakly interacting particles with a fixed total energy. Each particle has allowed energies ϵ_j $(j = 0, 1, 2, \ldots)$. Write down the Boltzmann probability distribution for the probability $P(\epsilon_j)$ of a particle having energy ϵ_j, when the system is in equilibrium at temperature T.

Show that the average energy of any particle in this system is given by

$$\overline{\epsilon} = \frac{\partial}{\partial \beta} \ln Z,$$

where Z is the partition function, which you should define, and $\beta = -\frac{1}{kT}$, where k is Boltzman's constant. Show also that the variance of the particle energy, defined by

$$\sigma_\epsilon^2 = \overline{\epsilon^2} - (\overline{\epsilon})^2 \, ,$$

can be written

$$\sigma_\epsilon^2 = \frac{\partial^2}{\partial \beta^2} \ln Z.$$

The allowed energy levels of a system are given by $\epsilon_j = \pm\epsilon, \pm 2\epsilon$. Show that

$$Z = 2(\cosh \beta\epsilon + \cosh 2\beta\epsilon),$$

and use your previous results to compute both $\overline{\epsilon}$ and σ_ϵ^2 as functions of $\frac{\epsilon}{kT}$.

Give the high and low temperature limits of $\overline{\epsilon}$ in this case. [20 marks]

6. An Ising-like model with 4 sites has energy

$$E(\{s\}) = -\sum_{m=1}^{4} J_m s_m s_{m+1},$$

where $s_{m+4} = s_m$, $s_m = \pm 1$ and the bond strength J_m satisfies

$$J_m = \begin{cases} J & \text{if } m \neq 4, \\ -J & \text{if } m = 4, \end{cases}$$

where J is a positive constant.

Identify the possible microstates and hence show that in thermal equilibrium at temperature T, the partition function Z is given by

$$Z = 16 \cosh\left(\frac{2J}{kT}\right).$$

Find expressions for:

(a) the average energy $\overline{E}$;

(b) the probability that the system is in the state $(+1, -1, +1, -1)$;

(c) the correlations $\overline{s_2 s_4}$ and $\overline{s_3 s_4}$.

Show that at high temperatures

$$\overline{E} \approx -\frac{4J^2}{kT}.$$

[20 marks]

7. The bonds J_{ij} of a Hopfield model of a neural network consisting of N neurons $(i = 1, 2, \ldots N)$ are "trained" to memorise p patterns $\xi^{(r)}$, $r = 1 \ldots p$, using the Hebb rule

$$J_{ij} = J \sum_{r=1}^{p} \xi_i^{(r)} \xi_j^{(r)},$$

where J is a positive constant. Describe briefly the Hopfield algorithm and how it may be used to recall these patterns under certain circumstances.

Consider a Hopfield model with 5 neurons which has been trained using the Hebb rule with two patterns

$$\xi^{(1)} = (+1, -1, +1, +1, +1), \quad \xi^{(2)} = (-1, +1, +1, -1, +1).$$

If the network is initially in the state $s_0 = (-1, -1, +1, +1, +1)$, evaluate the relevant neural input sums, and the initial transition probabilities for each node. Hence compute the probability that one of the stored patterns is fully recalled in a single step. [20 marks]