MATH327 - Stochastic Processes and Statistical Mechanics
January 2001

Time allowed: Two hours and a half

Full marks can be obtained for complete answers to FIVE questions.
Only the best FIVE answers will be counted.

1. Define the term Markovian. with respect to a discrete time stochastic process.

A discrete time Markov process has three states denoted 1, 2 and 3, and the transitions between them have probabilities

$$P(1 \to 2) = \alpha, \quad P(2 \to 1) = \frac{1}{2}\alpha$$
$$P(2 \to 3) = \frac{1}{2}\alpha, \quad P(3 \to 2) = \alpha$$
$$P(3 \to 1) = 0, \quad P(1 \to 3) = 0\,,$$

where $0 < \alpha < 1$. Consider the resulting Markov chain and obtain the stochastic matrix Q.

Find the eigenvalues and *right* eigenvectors of Q and obtain the probability distribution as the number of steps $n \to \infty$.

The system is initially in state 1. Write down a vector describing the initial probability of the three-state system. Expand this vector with respect to the three right eigenvectors as a basis.

If $\alpha = \frac{1}{3}$, show that the probability to be in state 3 after n steps is

$$P(3,n) = \frac{1}{4} - \frac{1}{2}\left(\frac{2}{3}\right)^n + \frac{1}{4}\left(\frac{1}{3}\right)^n.$$

[20 marks]

2. For a stochastic process, define the probability transition rate $W(x \to x')$, $x \neq x'$ where x and x' are possible states of the process.

The possible states of a Markov process $X(t)$ are the positive integers. The transition rates are

$$W(x \to x') = \begin{cases} ax & x' = x+1, \\ bx & x' = x-2, \\ 0 & \text{otherwise}, \end{cases}$$

where a and b are positive constants (with $a \neq 2b$).

Write down the Master Equation for this process and hence show that

$$\overline{X(t)} = \overline{X(0)}e^{(a-2b)t}.$$

Also find an expression for $\frac{d\overline{X^2(t)}}{dt}$ in terms of $\overline{X^2(t)}$ and $\overline{X(t)}$.

Hence find a general expression for $\overline{X^2(t)}$. [20 marks]

3. The Master Equation for a particular class of Markovian process can be wriiten as

$$\frac{\partial}{\partial t}P(x,t) = \int_{-\infty}^{\infty} dr[P(x-r,t) - P(x,t)]f(r),$$

where the transition rate f is given by

$$f(r) = Ae^{-\frac{r^2}{b^2}},$$

and A and b are positive constants. Here, the stochastic variable can take any real value x, and $P(x,t)$ is the probability density function.

Show that, with suitable approximations, which you should explain, the Master Equation reduces to the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2}D\frac{\partial^2 P}{\partial x^2},$$

where you should find D in terms of A and b.

Verify that the probability density function

$$P(x,t) = \frac{c}{\sqrt{t}}e^{-\frac{\lambda x^2}{t}}$$

satisfies this equation, providing that the constants c and λ take on particular values which you should specify in terms of A and b.

Note: you may make use of the integrals

$$\int_{-\infty}^{\infty} dx e^{-a^2x^2} = \frac{\sqrt{\pi}}{a}$$

$$\int_{-\infty}^{\infty} dx x^2 e^{-a^2x^2} = \frac{\sqrt{\pi}}{2a^3}.$$

[20 marks]

4. The stochastic variable X satisfies the stochastic differential equation

$$\frac{dX}{dt} = -\frac{2X}{t} + A(t),$$

where the stochastic quantity A satisfies

$$\overline{A(t)} = \alpha, \quad \text{and} \quad \overline{A(t)A(t')} = \alpha^2 + \alpha\delta(t-t'),$$

where α is a positive constant. Use an integrating factor technique or other means to show that the solution for $X(t)$ satisfying $X(1) = 0$ is such that

$$\overline{X(t)} = \frac{\alpha}{3t^2}(t^3 - 1),$$

and find a corresponding expression for

$$\sigma^2(t) = \overline{X^2(t)} - \left(\overline{X(t)}\right)^2.$$

Verify that, for large t,

$$\frac{\overline{X(t)}}{\sigma^2(t)} \approx \frac{5}{3}.$$

[20 marks]

5. A system consists of a large number N of distinguishable, weakly interacting particles with a fixed total energy. Each particle has allowed energies ϵ_j $(j = 0, 1, 2, \ldots)$. Write down the Boltzmann probability distribution for the probability $P(\epsilon_j)$ of a particle having energy ϵ_j, when the system is in equilibrium at temperature T.

Show that the average energy of any particle in this system is given by

$$\bar{\epsilon} = \frac{\partial}{\partial \beta} \ln Z,$$

where Z is the partition function, which you should define, and $\beta = -\frac{1}{kT}$, where k is Boltzman's constant. Show also that the variance of the particle energy, defined by

$$\sigma_\epsilon^2 = \overline{\epsilon^2} - (\bar{\epsilon})^2,$$

can be written

$$\sigma_\epsilon^2 = \frac{\partial^2}{\partial \beta^2} \ln Z.$$

The allowed energy levels of a system are given by $\epsilon_j = (2j+1)\epsilon$. Show that

$$Z = \frac{1}{2 \sinh\left(\frac{\epsilon}{kT}\right)}.$$

Use your previous results to compute both $\bar{\epsilon}$ and σ_ϵ^2 as functions of $\frac{\epsilon}{kT}$.

Hint: You may wish to use the formula for the sum of a geometric series.

[20 marks]

6. An Ising-like model with 4 sites has energy

$$E(\{s\}) = -J \sum_{m=1}^{4} s_m s_{m+1} + \frac{1}{4} J \sum_{m=1}^{4} s_m s_{m+2},$$

where $s_{m+4} = s_m$ and $s_m = \pm 1$. Here J is a positive constant.

Identify the possible microstates and hence show that in thermal equilibrium at temperature T, the partition function Z is given by

$$Z = 2\left[e^{3\kappa} + e^{-5\kappa} + 4 + 2e^{\kappa}\right],$$

where you should identify κ.

Find expressions for the average energy $\overline{E}$ and the probability that the system is in the state $(+1, +1, -1, -1)$.

Compute the correlations $\overline{s_1 s_2}$ and $\overline{s_1 s_3}$.

[20 marks]

7. The bonds J_{ij} of a Hopfield model of a neural network consisting of N neurons $(i = 1, 2, \ldots N)$ are "trained" to memorise p patterns $\xi^{(r)}$, $r = 1 \ldots p$, using the Hebb rule

$$J_{ij} = J \sum_{r=1}^{p} \xi_i^{(r)} \xi_j^{(r)},$$

where J is a positive constant. Describe briefly the Hopfield algorithm and how it may be used to recall these patterns under certain circumstances.

State the relationship of this model to a statistical mechanics system with energy given by

$$E(\{s\}) = -\sum_{i \neq j} J_{ij} s_i s_j \qquad (s = \pm 1),$$

in equilibrium at temperature T.

Consider a Hopfield model with 4 neurons which has been trained using the Hebb rule with two patterns

$$\xi^{(1)} = (+1, +1, -1, +1), \quad \xi^{(2)} = (+1, -1, +1, +1).$$

If the network is initialy in the state $s_0 = (+1, +1, -1, -1)$, evaluate the relevant neural input sums, and the initial transition probabilities for each node. Hence compute the probability that one of the stored patterns is fully recalled in a single step. [20 marks]