1996

2MA67 (=M327)

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. A small restaurant business fluctuates in successive years between three financial states as follows: 1 (bankruptcy), 2 (verge of bankruptcy) and 3 (solvency). The stochastic matrix giving the transition probabilities for evolving from state to state is

$$Q = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 \end{pmatrix}$$

where Q_{fi} is the transition probability for passing from state i to state f after one year's trading.

Comment on any assumptions being made when modelling this process as a Markov chain and illustrate your answer by giving examples of effects which would make the process non-Markovian.

Show that the eigenvalues (λ_i) of Q are $1, \frac{1}{2}$ and 0.

Find the corresponding right and left eigenvectors ($\mathbf{s}_i$ and $\tilde{\mathbf{t}}_i$ respectively) normalised so that

$$\tilde{\mathbf{t}}_i \mathbf{s}_j = \delta_{ij} \; .$$

Hence obtain an expression of the form

$$\mathbf{P}(n) = \sum_{i=1}^{3} \alpha_i \lambda_i^n \mathbf{s}_i$$

for the state probability vector after n years where you should determine the coefficients α_i in terms of the initial probability vector $\mathbf{P}(0)$.

Given that the restaurant starts out (at $n = 0$) on the verge of bankruptcy, use the above result to estimate how many years pass before the probability of being in business drops below 10%.

2. In a continuous time Markovian stochastic process, the probability that the stochastic variable X has some value x at time t is given by $P(x,t)$ and the transition probability rate for transitions $x \to x'$ at time t is denoted by $W(x \to x', t)$.

Write down the master equation for this process.

In a charity basketball record attempt, a hot-shot player is trying to score the maximum number of successful shots at basket in one hour. At any moment in time t, he is on a roll of $x(t)$ successive successful shots. Modelling this as a Markovian process, and taking X as the corresponding stochastic variable, the transition probability rate is given as

$$W(x \to x') = \begin{cases} s & \text{if } x' = x+1, \\ f & \text{if } x' = 0\ , \\ 0 & \text{otherwise,} \end{cases}$$

where s and f are positive constants.

Make a rough sketch of how you might expect the stochastic process $X(t)$ to behave.

Show that the master equation for this process can be written as

$$\frac{\partial}{\partial t} P(x,t) = -(s+f)P(x,t) + \begin{cases} f & \text{if } x = 0, \\ sP(x-1,t) & \text{if } x \geq 1\,. \end{cases}$$

Hence obtain an expression for the expected (i.e. average) number of consecutive successful shots at basket that the hot-shot player will have made at time t (assuming, of course, that he starts with no successful shots to his credit).

Given that the player is able to make 360 attempts in the hour, propose reasonable values for s and f (in units of seconds^{-1}). Justify your particular choice in relation to the supposed ability of the player.

Hence evaluate and make a rough sketch of the quantity $\overline{x}(t)$.

3. Show that the master equation for a continuous time Markovian process in which the transition probability rate between discrete values x and x' is a function of x' only, can be written as

$$\frac{\partial}{\partial t}P(x,t) = f(x) - \frac{1}{\tau}P(x,t)\,,$$

where you should define $f(x)$ and τ.

Hence find an expression for the probability $P(x,t)$, given some initial probability distribution $P_0(x)$, and determine the equilibrium probability distribution in terms of $f(x)$.

For a particular Markov process, x takes on non-negative integer values and

$$f(x) = a^x \quad (0 < a < 1)\,.$$

Determine the quantity τ and equilibrium probability distribution for this process.

For the case $a = 1/2$, make a rough sketch of this equilibrium probability distribution.

Also for $a = 1/2$, estimate the probability that $x(t) = 9$ after a long time (t very much greater than $1/2$) for the two cases

$$\text{(a)}\quad x(0) = 0 \qquad \text{and} \qquad \text{(b)}\quad x(0) = 9\,.$$

4. The number of cells y at time t in a particular type of organism is modelled by the stochastic differential equation

$$\frac{dy}{dt} = \lambda y + A(t)$$

where λ is a positive constant and $A(t)$ is a stochastic noise function satisfying

$$\overline{A(t)} = 0\,, \quad \text{and} \quad \overline{A(t_1)A(t_2)} = \Gamma\delta(t_1 - t_2)\,.$$

Here, Γ is another positive constant.

Given that a single cell exists at time $t = 0$, show that the average number of cells after time t is given by

$$\overline{y}(t) = e^{\lambda t}\,.$$

By considering $(y - \overline{y})^2$ or otherwise, find an expression for the variance

$$\sigma^2 = \overline{y^2} - \overline{y}^2$$

at time t.

Show that at large time, $\sigma/\overline{y}$ tends to a constant and find this constant.

If instead there were zero cells at time $t = 0$, find the expected mean and root mean square number of cells after time t.

5. A system consists of a large number N of distinguishable weakly interacting particles with a fixed total energy. Each particle has allowed energies ϵ_j $(j = 0, 1, 2\ldots)$. *Write down* the Boltzmann probability distribution governing the number of particles n_j of energy ϵ_j in such a system in equilibrium at temperature T.

Show that the average energy of any particle in this system is given by

$$\overline{\epsilon} = \frac{\partial}{\partial\beta} \ln Z$$

where Z is the partition function, which you should define, and β is a function of temperature and Boltzmann's constant k.

Each particle in such a system can be in one of three states having energies $-\epsilon$, 0 and ϵ.

Show that

$$Z = 1 + 2\cosh(\epsilon/kT)$$

and find the average energy per particle $\overline{\epsilon}$.

Verify that, at low temperature, $\overline{\epsilon}$ is as you would expect.

Describe the key difference between microcanonical and canonical ensembles.

For a system of particles such as the above in the canonical ensemble, the specific heat per particle is defined as

$$C = \frac{1}{N}\frac{\partial U}{\partial T}$$

where U is the total energy carried by all N particles.

Find the specific heat per particle for the above system of particles and verify that, at high temperature, it is approximately

$$\frac{2}{3}\frac{\epsilon^2}{kT^2}\,.$$

6. An Ising-like model with 3 sites has energy function

$$E(\{s\}) = -\sum_{m=1}^{3} J_{m,m+1} s_m s_{m+1}$$

where $m + 3 \equiv m$ and $s_m = \pm 1$. The inter-spin coupling $J_{m,m+1}$ is given by

$$J_{m,m+1} = \begin{cases} J & \text{if } m = 1 \text{ or } 2, \\ rJ & \text{if } m = 3, \end{cases}$$

where J and r are constants.

Identify the possible microstates and hence show that in a thermal equilibrium at temperature T, the partition function Z is given by

$$4e^{-r\kappa} + 4e^{r\kappa}\cosh(2\kappa)$$

where you should identify κ.

Find expressions for the average energy $\overline{E}$ and the probability that the system is in the state $(+1, +1, +1)$.

Sketch the energy of each class of microstate as a function of r and hence identify the ground state (or states) in the cases (a) $r < -1$ and (b) $r > -1$.

7. A Hopfield model of a neural network consists of N nodes carrying 'spins' $s_i = \pm 1$ $(i = 1, 2 \dots N)$ coupled together by bonds of strength J_{ij}. The firing probability function for node i is

$$P(s_i = \mu) = \frac{e^{\mu b V_i}}{e^{\mu b V_i} + e^{-\mu b V_i}}$$

where $\mu = \pm 1$, b is some positive constant and V_i is the node input sum.

Write down an expression for the node input sum in terms of the node 'spins' and the bonds.

State the relationship of the model with a statistical mechanics system in equilibrium at temperature T and whose energy function is given by

$$E(\{s\}) = -\sum_{i \neq j} J_{ij}\ s_i s_j\,, \qquad (s = \pm 1)\,.$$

Explain the stochastic process which leads to the recovery of stored 'memories'.

A Hopfield model with 3 nodes is 'trained' to memorise 2 patterns

$$\xi^{(1)} = (+1, +1, +1) \qquad \text{and} \quad \xi^{(2)} = (+1, -1, +1)\,,$$

using the Hebb rule

$$J_{ij} = J \sum_{r=1}^{2} \xi_i^{(r)} \xi_j^{(r)}\,.$$

Show that, in this case, the network behaves as two separate networks: one of two nodes and one of a single node.

The network starts from the initial state $s = (+1, -1, -1)$, and the Hopfield parameter b is chosen to have the value $3/J$. Find the probability that the network recovers the pattern $\xi^{(1)}$ in a single step and also the probability that it recovers $\xi^{(2)}$ in a single step.