1995 2MA67 (=M327)

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Each member of staff in a large company has an identical personal computer. The monitor (i.e. computer screen) is the most likely component to suffer failure. The breakdown and subsequent repair of a monitor is modelled as a two-state, discrete time, Markovian stochastic process with probability B over each day of a monitor failing and probability A over each day of a broken one being successfully repaired.

State the assumptions that are being made in this model and illustrate each assumption by giving one example, from a realistic application to the problem, that would invalidate that assumption.

Find the stochastic matrix describing the model. Given that initially all monitors are operational, show that the probability of a given monitor being operational after n days can be written as

$$(A + B\mu^n)/(A + B)$$

where you should determine μ.

The second most common component to fail in one of these computers is the hard disc. These fail with probability C (per day) and are successfully repaired with probability D (also per day). Assuming that

$$A = 1/2, \quad B = 1/500, \quad C = 1/4, \quad D = 1/1000 \qquad \text{(per day)},$$

estimate what percentage of the company's computers are *out of order* on a given day some long time after the initial set up.

Note any assumptions you have made.

2. In a continuous time Markovian stochastic process, the probability that the stochastic variable X has some value x at time t is given by $P(x,t)$ and the transition probability rate for transitions $x \to x'$ at time t is given by $W(x \to x', t)$.

Write down the master equation for this process.

The population X of a particular group of animals at time t is modelled as a Markov process. The transition rate between population values x and x' is given by

$$W(x \to x') = \begin{cases} ax & \text{if } x' = x+1, \\ ax/2 & \text{if } x' = x+2, \\ 0 & \text{otherwise,} \end{cases}$$

where a is a positive constant. Write down the master equation for this process and hence show that, with intial condition that $x = x_0$ when $t = 0$,

$$\overline{x}(t) = x_0 e^{2at} \,.$$

Obtain the corresponding value of $\overline{x^2}(t)$ and deduce that

$$\frac{\sigma(t)}{\overline{x}(t)} \approx \sqrt{\frac{3}{2x_0}}$$

when $t >> 1/a$ and where σ^2 is the variance.

If the initial population is 30 individuals and $a^{-1} = 1$ year, estimate the mean poulation and its variance after 3 years.

3. The master equation for a particular class of Markovian process can be written as

$$\frac{\partial}{\partial t}P(x,t) = \int_{-\infty}^{\infty} dr[P(x-r,t) - P(x,t)]f(r)\,,$$

where the transition rate f is given by

$$f(r) = A\exp(-r^2/b^2)$$

and A and b are positive constants. Here, the stochastic variable can take any real value x and $P(x,t)$ is the corresponding probability density function.

Show that with suitable approximations, which you should explain, the master equation reduces to the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{D}{2}\frac{\partial^2 P}{\partial x^2}\,,$$

where you should find D in terms of A and b.

Verify that the probability density function

$$P(x,t) = \frac{c}{\sqrt{t}}\exp(-\lambda x^2/t)$$

satisfies this equation provided that the constants c and λ take on particular values which you should specify in terms of A and b.

[*Note: you may make use of the integral*

$$\int_{-\infty}^{\infty} dx\exp(-a^2x^2) = \frac{\sqrt{\pi}}{a}$$

and others obtained from it by differentiation with respect to a^2.]

4. In a model of particles falling through air subject to gravity and resistive forces the vertical velocity v satisfies the Langevin equation

$$\frac{dv}{dt} + kv = F(t)$$

where k is some coefficient of air resistance. The stochastic force $F(t)$ satisfies

$$\overline{F(t)} = g\,, \quad \text{and} \quad \overline{F(t_1)F(t_2)} = g^2 + \Gamma\delta(t_1 - t_2)\,.$$

Here, the constant g models the acceleration due to gravity, which is vertically downwards, and the constant Γ represents the stochastic effects of collisions with air molecules which are in addition to the overall velocity-dependant resistance controlled by the constant k.

A number of particles are dropped with initial vertical velocity v_0. Show that the mean vertical velocity after time t is

$$v_0 e^{-kt} + \frac{g}{k}(1 - e^{-kt})\,.$$

Show also that the variance in the vertical velocity is proportional to

$$1 - e^{-2kt}$$

where you should find the constant of proportionality.

Assuming that the air is spatially homogeneous, write down the Langevin equations for the two *horizontal* components of velocity.

Hence write down the means and variances of these velocity components at time t, assuming that they are zero initially.

5. A system consists of a large number N of distinguishable weakly interacting particles with a fixed total energy. Each particle has allowed energies ϵ_j $(j = 0, 1, 2 \ldots)$. Write down the Boltzmann probability distribution governing this system in equilibrium at temperature T.

Show that the average energy of any particle in this system is given by

$$\bar{\epsilon} = \frac{\partial}{\partial \beta} \ln Z$$

where Z is the partition function, which you should define, and β is a function of temperature and Boltzmann's constant k.

The allowed energy levels of a system are given by $\epsilon_j = (2j+1)\epsilon$. Show that

$$Z^{-1} = \exp(\frac{\epsilon}{kt}) - \exp(-\frac{\epsilon}{kt})$$

and find the average energy per particle $\bar{\epsilon}$.

Find the high temperature $(T \to \infty)$ and low temperature $(T \to 0)$ limits of the average particle energy and make a rough sketch of this quantity.

[*Hint: you may wish to use the formula for the sum of a geometric series*]

6. An Ising-like model with 4 sites has energy

$$E(\{s\}) = -J \sum_{m=1}^{4} s_m s_{m+1} + rJ \sum_{m=1}^{4} s_m s_{m+2}$$

where $m + 1 \equiv m$ and $s_m = \pm 1$. Here J and r are positive constants.

Identify the possible microstates and hence show that in a thermal equilibrium at temperature T, the partition function Z is given by

$$e^{4(1-r)\kappa} + e^{-4(1+r)\kappa} + 4 + 2e^{4r\kappa} \,,$$

where you should identify κ.

Find expressions for the average energy $\overline{E}$ and the probability that the system is in the state $(+1, +1, +1, +1)$.

Sketch the energy of each class of microstate as a function of r and hence identify the ground state (or states) in the cases (a) $r < 1/2$ and (b) $r > 1/2$.

7. The bonds J_{ij} of a Hopfield model of a neural network consisting of N neurons $(i = 1, 2 \dots N)$ are 'trained' to memorise p patterns $\xi^{(r)}, r = 1 \dots p$. using the Hebb rule

$$J_{ij} = J \sum_{r=1}^{p} \xi_i^{(r)} \xi_j^{(r)} .$$

Describe briefly the Hopfield alogorithm and how it may be used to recall these patterns under certain circumstances.

State the relationship of this model to a statistical mechanics system with energy given by

$$E(\{s\}) = -\sum_{i \neq j} J_{ij} \; s_i s_j \,, \qquad (s = \pm 1)$$

in equilibrium at temperature T.

Consider a Hopfield model with 4 neurons which has been trained using the Hebb rule with two patterns

$$\xi^{(1)} = (+1, +1, +1, +1) \qquad \text{and} \quad \xi^{(2)} = (+1, -1, +1, -1) \,.$$

If the network is initially in the state $s_0 = (+1, -1, -1, -1)$, evaluate the relevant neural input sums and hence estimate the probability that one of the stored patterns is fully recalled in a single step.