1994

2MA67 (=M327)

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **SIX** answers will be taken into account.

1. A discrete time Markovian process involving a discrete stochastic variable is described by a Markov chain

$$P'_m = \sum_n Q_{mn} P_n \, .$$

State

(i) a property of the stochastic matrix $\mathbf{Q}$ which applies to each of its elements

(ii) a property of $\mathbf{Q}$ which applies to each of its columns

(iii) the largest eigenvalue of $\mathbf{Q}$ and its corresponding left eigenvalue.

On any one day, each vehicle of a car rental company can be thought of as being in one of three states: (1) ready and waiting for hire (2) out on hire or (3) in repair/preparation for rehire. The probability (per day) that a car be hired is $\frac{1}{4}$; that a car on hire be returned is $\frac{1}{8}$; and that a returned car be made ready for rehire is $\frac{3}{4}$. Show how the system can be described by a Markov chain with stochastic matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{7}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

where the possible states are ordered as above. Comment very briefly on simplifying assumptions which are being made and on real effects which are being ignored.

Find the eigenvalues and right eigenvectors of $\mathbf{Q}$.

The company opens for business with a large fleet of cars all ready for hire and operates as described above for a large number of days. Find the equilibrium percentages of cars in the 3 categories: awaiting hire, out on hire and being prepared.

2. The master equation for a Markovian stochastic process can be written

$$\frac{\partial P}{\partial t}(x,t) = \sum_{x'} [P(x',t)W(x' \to x,t) - P(x,t)W(x \to x',t)] \, .$$

Describe briefly the significance of the quantities $P(x,t)$ and $W(x \to x',t)$.

Frustrated workers attending to a sequence of tasks are sometimes heard to say "It's a case of one step forwards and two steps back!". Consider a model of such progress in which the steps are labelled by x, an integer, and where the forward steps $x \to x+1$ are made with probability rate a and the backwards steps $x \to x-2$ with rate λa where a and λ are positive constants.

Write down a master equation for this stochastic process.

Evaluate $\overline{x(t)}$ and $\overline{x^2(t)}$ given that the process starts at step 0 at time $t = 0$.

Find the value of λ for which, on average, no progress is made and in this case find the root mean square (RMS) deviation from the starting point after time t.

In a model of progress in which $\lambda = 1$, find the average rate of progress and the corresponding RMS deviation.

Describe, with the aid of a sketch, the evolution with time of the distribution in x for the model with arbitrary λ.

3. The quantity H is defined for a Markov process as

$$H(t) = \sum_{x} P(x,t) \ln \frac{P(x,t)}{P_e(x)} \, ,$$

where $P_e(x)$ is the equilibrium probability distribution. State, without proof, the 'H Theorem'.

A Markov process has two possible states 1 and 2 with transition rates given by

$$W(1 \to 2) = \lambda \quad \text{and} \quad W(2 \to 1) = \mu \, ,$$

where λ and μ are positive constants. Write down the corresponding master equation.

Initially, the probability that $x = 1$ is a. Show that

$$P(1,t) = \frac{1}{b}\left(\mu - f e^{-bt}\right) ,$$

where f and b are constants which you should determine.

Use this result to verify that the H Theorem is valid for this process by evaluating dH/dt.

If $a = \frac{1}{2}$ and $\lambda = \frac{\mu}{2}$, show that, at $t = 0$, $\dot{H} = -\frac{\mu}{4} \ln 2$.

4. The real variable y satisfies the ordinary differential equation

$$\frac{dy}{dt} = -\frac{y}{t} + \alpha$$

whereas the stochastic variable x satisfies the stochastic differential equation

$$\frac{dx}{dt} = -\frac{x}{t} + A(t)$$

where α is a positive constant and the stochastic quantity $A(t)$ satisfies

$$\overline{A(t)} = \alpha, \quad \text{and} \quad \overline{A(t)A(t')} = \alpha^2 + \alpha\delta(t - t')\,.$$

Use an integrating factor technique or other means to find the solution for $y(t)$ satisfying $y(1) = 0$.

Show that the solution for $x(t)$ satisfying $x(1) = 0$ is such that

$$\overline{x(t)} = \frac{\alpha}{2t}(t^2 - 1)$$

and find a corresponding expression for

$$\sigma^2(t) = \overline{x^2(t)} - \left(\overline{x(t)}\right)^2$$

.

Verify that, for large t,

$$\overline{x(t)}/\sigma^2 \approx \frac{3}{2}\,.$$

Give one example of a simple physical system which can be described by a stochastic differential equation. You should give and describe the relevant equation but need not solve it.

5. A system consists of a large number N of distinguishable weakly interacting particles and has total energy U. If each particle has allowed energies ϵ_j ($j = 1, 2 \ldots M$), derive the probability that a particle is in a particular one of these energy levels.

[You may use the result that $\log N! \approx N \log N - N$ for large N.]

Write down the constraint equations for the total energy and number of particles for two such systems in thermal equilibrium. Use these constraints to derive the Boltzmann distribution for the energy of a particle in the above system of N particles when they are in thermal equilibrium at temperature T.

Show that the average energy of any particle in this system is given by

$$\bar{\epsilon} = \frac{\partial}{\partial \beta} \ln Z$$

where Z is the partition function, which you should define, and β is a function of temperature and Boltzmann's constant k.

Consider such a system with three allowed energy levels given by $\epsilon_j = 0, \pm\epsilon$. Evaluate Z, and the average energy per particle $\bar{\epsilon}$.

Give the high and low temperature limits of the average particle energy for this three level system.

6. An Ising model with 4 sites has energy

$$E(\{s\}) = -\sum_{m=1}^{4} J_m \; s_m s_{m+1}$$

where $m + 4 \equiv m$, $s_m = \pm 1$ and the bond strength J_m satisfies

$$J_m = \begin{cases} J & \text{if } m \neq 4, \\ -J & \text{if } m = 4, \end{cases}$$

where J is a positive constant.

Show that in a thermal equilibrium at temperature T, the partition function Z is given by

$$16 \cosh(2J/kT)\,.$$

Find the average energy $\overline{E}$ and the probability that the system is in the state $(+1, +1, +1, +1)$.

Identify the ground state (or states).

Make a rough sketch of the average energy as a function of $1/T$, indicating the high and low temperature limits.

Show that at high temperatures

$$\overline{E} \approx -\frac{4J^2}{kT}\,.$$

7. Specify mathematically the Hopfield model of a neural network and describe briefly how it may be used to model the mechanisms of learning and of memory retrieval.

State the relationship of this model to a statistical mechanics system with energy given by

$$E(\{s\}) = -\sum_{i \neq j} J_{ij} \; s_i s_j$$

in equilibrium at temperature T.

Consider a Hopfield model with 5 neurons which has been trained using the Hebb rule with one pattern $\xi = (+1, +1, +1, -1, -1)$. If the network is initially in the state $s_0 = (+1, +1, -1, -1, -1)$, discuss the probability for the next state of the network.