1993

2MA67 (=M327)

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **SIX** answers will be taken into account.

1. With respect to a discrete time stochastic process, define the term Markovian.

A discrete time Markov process has three states, denoted 1, 2 and 3, and the transitions between them have probabilities

$$\begin{array}{ll} P(1 \to 2) = a & P(2 \to 1) = 0 \\ P(2 \to 3) = b & P(3 \to 2) = 0 \\ P(3 \to 1) = 0 & P(1 \to 3) = 0 \end{array}$$

with $0 < a, b < 1$. Consider the resulting Markov chain and obtain the stochastic matrix Q.

Find the right eigenvectors and eigenvalues of Q and obtain the probability distribution as the number of steps $n \to \infty$.

If the system is initially in state 1, show that the probability to be in state 3 after n steps (with $a = 1/3$ and $b = 2/3$)is

$$P(3, n) = 1 + (\frac{1}{3})^n - 2(\frac{2}{3})^n$$

2. For a stochastic process, define the probability transition rate $W(x \to x')$, $x \neq x'$, where x and x' are possible states of the process.

The possible states of a Markov process are the positive integers. The transition rates are

$$W(x \to x') = \begin{cases} gx & x' = x + 1 \\ px & x' = x + 2 \\ 0 & \text{otherwise,} \end{cases}$$

where g and p are positive constants.

Write down the Master Equation for this process and hence show that

$$\overline{x(t)} = \overline{x(0)} \exp((g + 2p)t)$$

Also find an expression for $d\overline{x^2(t)}/dt$ in terms of $\overline{x^2(t)}$ and $\overline{x(t)}$.

Hence find the general expression for $\overline{x^2(t)}$.

3. Consider an asymmetric random walk with probability transition rates

$$W(x \to x') = \begin{cases} a & x' = x + 1 \\ b & x' = x - 1 \\ 0 & \text{otherwise,} \end{cases}$$

where x takes integer values.

Write down the master equation and find $\overline{x(t)}$ and $\overline{x^2(t)}$ given the initial condition $x(0) = 0$.

Show that, with an appropriate approximation (which you should specify) the master equation can be expressed as

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{D}{2}\frac{\partial^2 P}{\partial x^2}$$

Relate v and D to the constants a and b.

Show that this equation can be reduced to the diffusion equation by a change of variables to a moving frame of reference.

4. Consider a particle falling with velocity v in a viscous medium and subjected to a stochastic force $A(t)$

$$\frac{dv}{dt} = g - \gamma v + A(t),$$

where $\overline{A(t)} = 0$, and $\overline{A(t)A(t')} = \lambda\ \delta(t - t')$.

Evaluate $\overline{v(t)}$ and $\overline{v^2(t)}$ given the initial condition $v(0) = 0$.

Hence obtain the variance of the velocity in the large time limit.

Write down the probability distribution for v at large time.

5. Consider a system consisting of a large number N of distinguishable weakly interacting particles with total energy U. If each particle has allowed energies ϵ_j, derive the probability that a particle is in a particular energy level.

State how the energy and number of particles are constrained for two systems in thermal equilibrium. Use this to derive the Boltzmann distribution for the energy of a particle in the above system of N particles when they are in thermal equilibrium at temperature T.

Show that the entropy of this system is given by

$$S = kN \log Z + U/T$$

Consider such a system with two allowed energy levels given by $\epsilon_j = 0, \epsilon$. Evaluate Z, the average energy per particle $\bar{\epsilon}$, and S.

You may use the result that $\log N! \approx N \log N - N$ for large N.

6. Consider an Ising model with N sites and with energy

$$E(\{s\}) = -\sum_{m=1}^{N} J\ s_m s_{m+1}$$

where $m + N \equiv m$ and $s_m = \pm 1$. Show that in thermal equilibrium at temperature T, the partition function is given by

$$Z = (2\cosh K)^N + (2\sinh K)^N, \text{ where } K = J/kT.$$

At large N, where $(\tanh K)^N$ can be neglected, evaluate $\overline{E}$ and hence, or otherwise, evaluate $\overline{s_m s_{m+1}}$. Also obtain an expression for $d\overline{E}/dT$.

7. Describe briefly the Hopfield model of a neural network, including a discussion of a learning mechanism and of memory retrieval.

State the relationship of this model to a statistical mechanics system with energy given by

$$E(\{s\}) = -\sum_{i \neq j} J_{ij}\ s_i s_j$$

at equilibrium at temperature T.

A Hopfield neural network with N neurons is trained using the Hebb rule with one memory ξ_i. Find the minimum energy configurations of the model. Evaluate the energy gap from these minimum energy configurations to the next energy level.