PAPER CODE NO.
MATH326

THE UNIVERSITY
of LIVERPOOL

SUMMER 2007 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Physics : Year 4
No qualification aimed for : Year 1

RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

The following results may be used freely as required

$$\Gamma^{\mu}_{\ \alpha\beta} = g^{\mu\nu}\Gamma_{\nu\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}_{\ \nu\sigma\rho} = \Gamma^{\mu}_{\ \nu\rho,\sigma} - \Gamma^{\mu}_{\ \nu\sigma,\rho} + \Gamma^{\mu}_{\ \alpha\sigma}\Gamma^{\alpha}_{\ \nu\rho} - \Gamma^{\mu}_{\ \alpha\rho}\Gamma^{\alpha}_{\ \nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}_{\ \mu\sigma\nu} \quad , \quad R = R^{\mu}_{\ \mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$c = 3\cdot 10^{8}\ \mathrm{ms}^{-1}$$

1. **(a)** State the two basic principles of special relativity.

(b) Consider two inertial frames I and I' in standard configuration: I and I' coincide at $t = t' = 0$, and I' moves with speed V relative to I in positive x-direction. Follow the steps below to derive the Lorentz transformation of x and t from I to I', assuming that this transformation is independent of y and z.

Write down the general linear transformation from the coordinates (x, t) of an event in I to those in another frame I', (x', t'), and apply the constraints due to the above configuration. Use the symmetry to do the same for the inverse transformation $x(x', t')$.

Determine the remaining unknown in $x' = x'(x, t)$ by applying the second principle of special relativity for light emitted at $x = 0$, $t = 0$ to this transformation and the inverse $x = x(x', t')$.

Finally use these two relations to specify also the second part $t' = t'(x, t)$ of the transformation from I to I'.

(c) Check your results of (b) by showing that they can be written in the form

$$\begin{aligned} x' &= \;\; x \cosh\phi - ct \sinh\phi \\ ct' &= -x \sinh\phi + ct \cosh\phi \end{aligned}$$

with $\tanh\phi = V/c$ and $\cosh\phi = (1 - V^2/c^2)^{-1/2} \equiv \gamma(V)$.

[**20 marks**]

2. **(a)** Employ the standard-configuration Lorentz transformation as given in problem 1.(c) to derive the relativistic addition law of collinear velocities

$$v'_x \equiv \frac{dx'}{dt'} = \frac{v_x - V}{1 - v_x V/c^2} \quad \text{with} \quad v_x \equiv \frac{dx}{dt} \;.$$

(b) The speed of light in a medium is $c_m = c/n$ in the rest frame of the medium, where n is its index of refraction and c the speed of light in vacuum.

Derive an approximation, including only terms not suppressed by powers of V/c, for the speed of a light beam in the medium as measured by an inertial observer moving with speed V relative to the medium in the direction opposite to that of the light.

Check your result by discussing the limiting cases $n = 1$ and $n \gg 1$. For a specific medium, the above observer measures the speed of the light beam as $v = 13/18\,c$ for $V = 0.1\,c$. Use the approximation derived above to determine the refractive index n of that medium.

(c) An ideal clock moves with the time-dependent speed $v(t)$ relative to an inertial frame I. Write down the general relation between the time τ shown by this clock and the time t in I, if $\tau = 0$ for $t = 0$. Show that for the velocity

$$v(t) = \alpha t \left(1 + \frac{\alpha^2 t^2}{c^2}\right)^{-1/2}$$

with $\alpha = \text{const.}$ the explicit relation between τ and t reads

$$t = \frac{c}{\alpha} \sinh\left(\frac{\alpha\tau}{c}\right) \;.$$

Determine t for $\alpha = 30\ \text{ms}^{-2}$ and $\tau = 6 \cdot 10^7\,\text{s}$.

[**20 marks**]

3. **(a)** Define the 4-velocity u^μ and express it in terms of γ, c and $\vec{v} \equiv d\vec{x}/dt$. Calculate $u_\mu u^\mu$. Write down the 4-momentum p^μ of a massive particle in terms of its mass m and 4-velocity u^μ. Identify the components of p^μ in terms of energy and 3-momentum $\vec{p}$, and derive the relation between energy, 3-momentum and mass of the particle.

(b) A stationary particle of mass m is struck by a particle of the same mass and energy E moving in the positive x-direction. Two particles, each of mass $2m$ result from this collision. One of these moves off at the angle θ to the positive x-direction, the other at the $-\theta$, where $\theta = 0$ corresponds to motion along the positive x-direction.

Employ the conservation of 3-momentum and energy to show that that the two particles emerging from the collision have the same energy, and to derive (in units with $c = 1$)

$$\cos^2\theta = \frac{E^2 - m^2}{E^2 + 2Em - 15m^2} .$$

Calculate the minimal energy E_{min} required for this process to take place.

(c) For $E = E_{\text{min}}$, determine the velocity $\vec{v}\,'$ of the center-of-mass frame (CMS) of the particles relative to the frame used in (b). What is the minimal energy E'_{min} of the initial particles in this frame?

[**20 marks**]

4. (a) Demonstrate that the covariant curvature tensor $R_{\mu\nu\rho\sigma}$ is antisymmetric under the exchange of ρ and σ. State the other symmetry properties of $R_{\mu\nu\rho\sigma}$.

Use these symmetries to show that in a two-dimensional Riemann space all components of $R_{\mu\nu\rho\sigma}$ are either zero, $+R_{1212}$ or $-R_{1212}$.

(b) Consider a surface with coordinates x_1, x_2 and a diagonal metric tensor with $g_{11} = a^2$, $g_{22} = a^2 \cos^2 x_1$.

Calculate the only non-vanishing Christoffel symbols Γ^1_{22} and $\Gamma^2_{21} = \Gamma^2_{12}$, and employ the results to prove $R_{1212} = a^2 \cos^2 x_1$.

(c) Determine the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R for the surface and coordinates of (b). Show that the Einstein tensor $G_{\mu\nu}$ vanishes.

[**20 marks**]

5. **(a)** Write down the covariant derivatives $\phi_{;\nu}$ and $a_{\mu;\nu}$ of scalar fields ϕ and covariant vector fields a_μ in terms of ordinary derivatives and Christoffel symbols. Use the latter to derive the corresponding expression for the covariant derivatives $T_{\mu\nu;\rho}$ of (0,2) tensor fields $T_{\mu\nu}$.

(b) Check the last result of (a) by verifying the relation $g_{\mu\nu;\rho} = 0$ for the metric tensor $g_{\mu\nu}$. Explain in words why this relation has to be fulfilled in any Riemann space. For which class of tensor fields $A_{\mu\nu}$ is also $B_{\mu\nu\rho}$ a tensor field if

$$B_{\mu\nu\rho} \;=\; A_{\mu\nu,\rho} + A_{\rho\mu,\nu} + A_{\nu\rho,\mu} \quad ?$$

(c) Prove, by considering the quantity $a^\mu b_\mu$ for an arbitrary covariant vector b_μ, that the covariant derivative of a contravariant vector a^μ is given by

$$a^\mu{}_{;\nu} \;=\; a^\mu{}_{,\nu} + \Gamma^\mu{}_{\sigma\nu} a^\sigma \ .$$

State, with an explanation but without proof, the corresponding result for the covariant derivative of a (2,1) tensor $T^{\mu\nu}{}_\rho$.

[**20 marks**]

6. The metric for a Schwarzschild spacetime can be written as

$$ds^2 \;=\; \left(1-\frac{2M}{r}\right)dt^2 \;-\; \left(1-\frac{2M}{r}\right)^{-1}dr^2 \;-\; r^2d\theta^2 \;-\; r^2\sin^2\theta\,d\phi^2$$

in units with $c=1$, where M is a constant parameter.

(a) Explain which components p_μ of the four-momentum are conserved, with respect to the coordinate system $x^\mu=(t,r,\theta,\phi)$, for a particle freely falling in this spacetime. Find the corresponding components of p^μ in terms of those of p_μ.

(b) A particle of mass m moves in the equatorial plane $\theta=\pi/2$ of this spacetime. Employ $p^\mu p_\mu=m^2$ to derive the effective potential $\widetilde{V}^2(r)$ in the equation

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; \widetilde{E}^2 \;-\; \widetilde{V}^2(r)$$

governing the radial motion. Use the notation $p_t=m\widetilde{E}$ and $p_\phi=m\widetilde{L}$.

(c) Derive the condition on $\widetilde{V}^2(r)$ for a circular orbit of a massive particle. Determine the radii for circular orbits for $M=1$ and $\widetilde{L}=4$. Sketch $\widetilde{V}^2(r)$ for these values of M and $\widetilde{L}$, and determine the range of $\widetilde{E}$ for pseudo-elliptic orbits. For which values of $\widetilde{L}$ are hyperbolic orbit possible if $M=1$?

[**20 marks**]

7. (a) Write down, with a brief explanation, the differential equation for the world line $z^\mu(\tau)$ of a freely falling massive particle. Show that in terms of the covariant derivative of the four-velocity $u^\mu = dz^\mu/d\tau$ this equation can be written as $u^\mu{}_{;\sigma}u^\sigma = 0$.

Use this relation to prove that the four-momentum component p_ρ is a constant along the trajectory of the particle, if the metric tensor $g_{\mu\nu}$ is independent of the coordinate x^ρ.

From now on consider the free motion of a particle with $p^i \ll p^0$ $(i = 1, 2, 3)$ in a spacetime with

$$ds^2 \;=\; (1+2\phi)\,dt^2 \;-\; (1-2\phi)\,[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

in units with $c = 1$, where $\phi \ll 1$ does not depend on t.

(b) Express the conserved momentum p_0 in terms of ϕ, $\vec{p}^{\,2} = p^i p^i$ and the mass m of the particle, neglecting all contributions beyond the first order in ϕ and $\vec{p}^{\,2}$. Interpret all terms of your result.

(c) Use the results of (a) to derive $dp^i/d\tau$ at the lowest non-trivial order in the limits $\phi \ll 1$ and $p^i \ll p^0$. Compare your result with the Newtonian equation of motion, $d^2\vec{z}/dt^2 = -\vec{\nabla}V$, where V represents the gravitational potential.

[20 marks]