PAPER CODE NO.
MATH326

THE UNIVERSITY
of LIVERPOOL

MAY 2006 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Physics : Year 4
No qualification aimed for : Year 1

RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R = R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}$$

$$\gamma(v) = 1/\sqrt{(1 - v^2/c^2)}$$

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1. (i) Write down one principle of special relativity.

(ii) An observer O' moves with speed $v = 2.5 \times 10^8$ ms^{-1} with respect to observer O along the positive x direction of O's reference frame. The O' observer measures the length of a rod at rest with respect to themselves, laid in their x direction, to be 10m. What length does the observer O measure the length of the same rod?

(iii) The inertial frame S' moves with constant relative speed v in the positive x direction of the inertial frame S. A particle moves with velocity $\mathbf{u} = (u_x, u_y, u_z)$ and acceleration $\mathbf{a} = (a_x, a_y, a_z)$ with respect to an observer in the inertial frame S. The same particle moves with velocity $\mathbf{u}' = (u'_x, u'_y, u'_z)$ and acceleration $\mathbf{a}' = (a'_x, a'_y, a'_z)$ with respect to an observer in the inertial frame S'. Use the Lorentz transformation between the S frame and S' to derive an expression for the differential dt' in terms of dt, dx and $\gamma(v)$. Hence, given the following expressions for the velocity $\mathbf{u}'$ of the particle in the S' frame in terms of the velocity of the particle in the S frame

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \quad , \quad u'_y = \frac{u_y}{\gamma(v)(1 - \frac{u_x v}{c^2})} \quad , \quad u'_z = \frac{u_z}{\gamma(v)(1 - \frac{u_x v}{c^2})} \ ,$$

derive expressions for the components a'_x and a'_y of the acceleration of the particle $\mathbf{a}'$ in the S' frame in terms of the components a_x and a_y of the acceleration $\mathbf{a}$ of the particle in the S frame.

(iv) In the frame S, two particle are sent out from the spatial origin at time $t = 0$ with speed v in different directions. The first particle moves in the positive x direction of the S frame. The second particle moves in the positive y direction of the S frame. Show that the magnitude of the velocity of one particle relative to the other is

$$v\left(2 - \frac{v^2}{c^2}\right)^{1/2} .$$

[20 marks]

2. Define the momentarily co-moving reference frame (MCRF).

A particle moves from rest at the origin of a frame S along the x-axis with constant acceleration α as measured in its MCRF. Given that the transformation of the acceleration in the x direction between the S inertial frame and the S' inertial frame (moving with velocity v in the positive x direction relative to the S frame) is

$$a'_x = \frac{a_x}{\gamma^3(v)(1 - \frac{v_x v}{c^2})^3} \, ,$$

show that the velocity after time t is

$$v\gamma(v) = \alpha t \, ,$$

where α is the proper acceleration. If the motion is in the *positive* x-direction, then show that

$$x \; = \; \frac{c^2}{\alpha}\left[\left(1 + \frac{\alpha^2 t^2}{c^2}\right)^{1/2} \; - \; 1\right] .$$

Plot the world line of the particle on a space-time diagram for positive x and t.

Compute the velocity of the particle that has infinite proper acceleration.

[**20 marks**]

3. (a) Write down the relation between the 4-momentum of a particle and its 4-velocity and rest mass. A particle of rest mass m_1 and speed v_1 hits a second particle, with rest mass m_2, that is at rest. The first particle is absorbed by the second particle. Find the rest mass of the new particle in the frame where the second particle was initially at rest.

(b) A photon with wavelength λ, momentum $\frac{h}{\lambda}$ and energy $\frac{hc}{\lambda}$ collides with a stationary electron of rest mass m. The photon is scattered at an angle θ from its original direction, with energy $\frac{hc}{\lambda'}$. Write down all the 4-momentum vectors for this process in terms of the energy and 3-momentum. Write down the equations for the conservation of energy and momentum using 4-vectors for this process. Hence show that

$$\lambda' - \lambda = \frac{2h}{mc}\sin^2\frac{\theta}{2} \, .$$

[**20 marks**]

4. For the line element

$$ds^2 = \frac{dx^2}{y^2} + \frac{dy^2}{y^2} \ ,$$

of a two dimensional surface with coordinates $x_1 = x$ and $x_2 = y$, compute all the Christoffel symbols.

Write down the general equation for a geodesic in terms of coordinates x^μ. Hence show that

$$x^{''} - \frac{2}{y} x^{'} y^{'} = 0 \ ,$$

$$y^{''} + \frac{1}{y}[(x^{'})^2 - (y^{'})^2] = 0 \ ,$$

where $x^{'} = \frac{dx}{d\tau}$ and $y^{'} = \frac{dy}{d\tau}$.

Define the covariant derivative of the 4-vector p_μ in terms of its ordinary derivative and the Christoffel symbols. Use the geodesic equation

$$p^\alpha p_{\beta;\alpha} = 0 \ ,$$

for 4-momentum to show that

$$m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\alpha p^\nu \ \ ,$$

for a particle of mass m. What consequence does the above equation have for a particle moving in a given metric?

[20 marks]

5. The line element for a two dimensional surface is

$$ds^2 \;=\; dv^2 \;-\; v^2 dw^2 \;,$$

with coordinates $x^\mu = (v, w)$. Write down the components of the metric tensor and its inverse. Show that the only non-zero components of the Christoffel symbol are

$$\Gamma^v_{ww} \;=\; v \quad , \quad \Gamma^w_{wv} \;=\; \frac{1}{v} \;.$$

Compute

$$R^w{}_{vwv} \;.$$

How many independent components are there of the Riemann curvature tensor in two dimensions? What does the value of $R^w{}_{vwv}$ imply about the geometry of the surface?

Find a coordinate transformation between the coordinates v, w to the coordinates x, t with line element

$$ds^2 = dx^2 - dt^2 \;.$$

[20 marks]

6. The metric for a Schwarzschild spacetime is given by

$$ds^2 \;=\; \left(1 - \frac{2M}{r}\right) dt^2 \;-\; \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \;-\; r^2 d\theta^2 \;-\; r^2 \sin^2\theta \, d\phi^2 \; ,$$

where $c = 1$ and M is constant in the coordinate system $x^\mu = (t, r, \theta, \phi)$.

Consider a particle of rest mass m moving in the equatorial plane $\theta = \pi/2$ of this spacetime. Assuming $p_t = m\tilde{E}$ and $p_\phi \;=\; -\,m\tilde{L}$ find all the components of p^μ. Use $p^\mu p_\mu = m^2$ to derive the effective potential $\tilde{V}^2(r)$ of the radial motion in the equation

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; \tilde{E}^2 \;-\; \tilde{V}^2(r) \; .$$

Write down the condition on $\tilde{V}^2(r)$ for a stable circular orbit. Hence show that the radius of a stable circular orbit R is:

$$R = \frac{\tilde{L}^2}{2M}\left(1 + \sqrt{1 - \frac{12M^2}{\tilde{L}^2}}\right) \; .$$

(You may assume that the circular orbit with the largest radius is stable.)

Show that the square of the energy of the particle in the stable orbit is

$$\tilde{E}^2 = \left(1 - \frac{2M}{R}\right)^2 \Big/ \left(1 - \frac{3M}{R}\right) \; .$$

Hence show that

$$\frac{dt}{d\phi} = \frac{dt}{d\tau}\Big/\frac{d\phi}{d\tau} = \left(\frac{R^3}{M}\right)^{1/2} \; ,$$

and derive the period of the particle in the stable circular orbit.

[20 marks]

7. The line element for a Schwarzschild spacetime is given by

$$ds^2 \;=\; \left(1-\frac{2M}{r}\right)dt^2 \;-\; \left(1-\frac{2M}{r}\right)^{-1}dr^2 \;-\; r^2 d\theta^2 \;-\; r^2\sin^2\theta\, d\phi^2$$

where $c = 1$. A massive particle moving in this spacetime obeys the following equation

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r)\ ,$$

$$\tilde{V}^2(r) = \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\ ,$$

where $\tilde{E} = \frac{p_t}{m}$ and $\tilde{L} = -\frac{p_\phi}{m}$ are defined in terms of the 4-momentum p_μ of the particle.

Consider a massive particle which falls radially from a distance R to the Schwarzschild radius at $2M$. Show that the relation between the infinitesimal proper time $d\tau$ and dr is

$$d\tau = -\frac{dr}{(\tilde{E}^2 - 1 + \frac{2M}{r})^{\frac{1}{2}}}\ .$$

When $\tilde{E} = 1$ show that the proper time for the particle to fall from radius R to the Schwarzschild radius at $2M$ is

$$\tau = \frac{4M}{3}\left[\left(\frac{R}{2M}\right)^{3/2} - 1\right]\ .$$

Derive the following relation between the infinitesimal proper and coordinate time for arbitrary $\tilde{E}$

$$dt = \frac{\tilde{E}}{(1-\frac{2M}{r})}d\tau\ .$$

The radial displacement from the Schwarzschild radius is defined to be $\epsilon = r - 2M$. Using $\tilde{E} = 1$, show that

$$dt = -\,\frac{(\epsilon + 2M)^{3/2}d\epsilon}{(2M)^{1/2}\epsilon}\ .$$

Comment on the finiteness of the proper time of the particle to fall from radius R to the Schwarzschild radius $2M$. Explain why an observer using time t will never see the particle pass over the Schwarzschild radius.

[20 marks]