PAPER CODE NO.
MATH326

THE UNIVERSITY
of LIVERPOOL

SUMMER 2005 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Physics : Year 4
No qualification aimed for : Year 1

RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R = R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}$$

$$\gamma(v) = 1/\sqrt{(1 - v^2/c^2)}$$

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1. Write down the Lorentz transformation relating the spacetime coordinates (x, t) in a frame S to those (x', t') in the frame S' moving with speed v in the positive direction along the positive x-axis in S. If $v = 2 \times 10^8$ ms^{-1}, what factor of increase does the observer in S measure the time periods of a clock at rest in the S' frame, compared to those timed by an observer at rest in S'?

A particle is moving with velocity $\mathbf{u} = (u_x, u_y, u_z)$ with respect to an observer in an inertial frame S. According to an observer in an inertial frame S' moving at constant relative velocity v in the positive x direction relative to S, the velocity of the same particle is $\mathbf{u}' = (u'_x, u'_y, u'_z)$. From the Lorentz transformation derive the following expressions

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \quad , \quad u'_y = \frac{u_y}{\gamma(v)(1 - \frac{u_x v}{c^2})} \; .$$

Two particles move away from the origin with velocities $\mathbf{v}$ and $-\,\mathbf{v}$ respectively where $v = \frac{3}{4}c$. Find the speed of one particle as measured by the other.

In the S frame, the particle moves with a velocity u at angle α to the positive x-axis. Show that the equivalent angle in the S' frame is given by

$$\tan \alpha' = \frac{\sin \alpha}{\gamma(v)(\cos \alpha - v/u)} \; .$$

[**20 marks**]

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2. A particle moves from rest at the origin of an inertial frame S along the positive x-axis with constant acceleration w as measured in a Momentarily Comoving Reference Frame (MCRF). Write down the corresponding differential equation for the velocity and solve it to obtain

$$v = \frac{wt}{\sqrt{1 + \frac{w^2 t^2}{c^2}}} \ .$$

Show that the proper time, defined by

$$\tau = \int_0^t \frac{1}{\gamma(v)} dt \ ,$$

of the particle is

$$\frac{c}{w} \sinh^{-1}\left(\frac{wt}{c}\right)$$

after time t in the lab frame.

Write down an expression for the 4-velocity, v_μ, of a particle with 3-velocity $(v, 0, 0)$. Demonstrate that $v_\mu v^\mu = c^2$. Hence, prove that the 4-acceleration defined by

$$w_\mu = \frac{dv_\mu}{d\tau} \ ,$$

where τ is the proper time, is orthogonal to the 4-velocity.

[**20 marks**]

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3. Two particles, called 1 and 2, with respective rest masses m_1 and m_2 collide. The collision does not change the masses, or internal state, of the particles. The initial energy and momentum of particle 1 are E_1 and p_1 and after the collision the energy and momentum of particle 1 are E_1' and p_1'. Similarly, for particle 2, the initial energy and momentum are E_2 and p_2 and the final energy and momentum are E_2' and p_2'.

Write down the conservation of energy and momentum using 4-vectors for this process. Compute $p_1^\mu p_{1\mu}$ in terms of the energy and 3-momentum of the particle 1, as well as its rest mass. Use conservation of momentum and energy to find an expression for ${p_2'}^\mu$ in terms of p_1^μ, p_2^μ, and ${p_1'}^\mu$. Hence show that

$$m_1^2 c^4 + p_1^\mu p_{2\mu} - p_1^\mu p_{1\mu}' - p_2^\mu p_{1\mu}' = 0 \ .$$

Now assume that particle 2 is at rest in the lab frame. Given that the angle between the p_1 and p_1' 3-momentum vectors is θ_1, show that

$$\cos\theta_1 = \frac{E_1'(E_1 + m_2 c^2) - E_1 m_2 c^2 - m_1^2 c^4}{p_1 p_1' c^2} \ ,$$

where p_1 and p_1' are the magnitude of initial and final 3-momentum for particle 1 respectively.

If particle 1 has zero mass ($m_1 = 0$), show that

$$E_1' = \frac{m_2 c^2}{1 - \cos\theta_1 + \frac{m_2 c^2}{E_1}} \ .$$

[20 marks]

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4. Define the covariant derivative of the vector a^μ in terms of its ordinary derivative and the Christoffel symbols. For the metric

$$g_{11} = (4 + \cos\phi)^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1$$

of a two dimensional surface with coordinates $x^1 = \theta$,$x^2 = \phi$, show that the non-vanishing Christoffel symbols are

$$\Gamma^1_{12} = -\frac{\sin\phi}{4 + \cos\phi} \quad , \quad \Gamma^2_{11} = (4 + \cos\phi)\sin\phi\ .$$

Hence, show the following components of the covariant derivative of $a^\mu = (\cos\phi, 0)$ are

$$a^1{}_{;2} = -\sin\phi\left(\frac{4 + 2\cos\phi}{4 + \cos\phi}\right) \quad , \quad a^2{}_{;1} = (4 + \cos\phi)\sin\phi\cos\phi\ .$$

Calculate $a_{1\,;2}$.

[20 marks]

5. The line element for a two dimensional surface is

$$ds^2 \;=\; A(x,y)dx^2 \;+\; B(x,y)dy^2 \; .$$

Write down the components of the metric tensor and its inverse. Show that

$$\Gamma^x_{xx} \;=\; \frac{1}{2A}\frac{\partial A}{\partial x} \quad \text{and} \quad \Gamma^x_{yy} \;=\; -\frac{1}{2A}\frac{\partial B}{\partial x} \; .$$

Given the remaining components of the Christoffel symbol (which you do not need to compute)

$$\Gamma^x_{xy} \;=\; \frac{1}{2A}\frac{\partial A}{\partial y} \;,\; \Gamma^y_{xx} \;=\; -\frac{1}{2B}\frac{\partial A}{\partial y} \;,\; \Gamma^y_{yy} \;=\; \frac{1}{2B}\frac{\partial B}{\partial y} \;,\; \Gamma^y_{xy} \;=\; \frac{1}{2B}\frac{\partial B}{\partial x} \; ,$$

show that

$$\begin{aligned} R^x{}_{yxy} \;&=\; -\frac{1}{2A}\left(\frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 B}{\partial x^2}\right) + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial x} + \left(\frac{\partial A}{\partial y}\right)^2\right) \\ &+\; \frac{1}{4AB}\left(\frac{\partial A}{\partial y}\frac{\partial B}{\partial y} + \left(\frac{\partial B}{\partial x}\right)^2\right) \; . \end{aligned}$$

The function D is defined by

$$D = R^x{}_{yxy} \; .$$

Use the symmetries of the Riemann curvature tensor to compute $R^y{}_{xyx}$ in terms of D and the components of the metric tensor.

[20 marks]

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6. The metric for a Schwarzschild spacetime is given by

$$ds^2 \;=\; \left(1-\frac{2M}{r}\right)dt^2 \;-\; \left(1-\frac{2M}{r}\right)^{-1} dr^2 \;-\; r^2 d\theta^2 \;-\; r^2\sin^2\theta\, d\phi^2 \;,$$

where $c = 1$ and M is constant in the coordinate system $x^\mu = (t, r, \theta, \phi)$. Explain which components of the four momentum $p_\mu = (p_t, p_r, p_\theta, p_\phi)$ of a particle are conserved. Find the components of $p^\mu = (p^t, p^r, p^\theta, p^\phi)$ in terms of those of p_μ.

Consider particle motion in the equatorial plane $\theta = \pi/2$ of this spacetime. For a particle of mass m, use $p^\mu p_\mu = m^2$ to derive the effective potential $\tilde{V}^2(r)$ of the radial motion in the equation

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; \tilde{E}^2 \;-\; \tilde{V}^2(r) \;,$$

where $p_t = m\tilde{E}$ and $p_\phi = m\tilde{L}$.

Write down the condition on $\tilde{V}^2(r)$ for a circular orbit for a massive particle. For $M = 0.5$ and angular momentum $\tilde{L} = 2$, find the radii for circular orbits. Determine which circular orbit is stable.

[**20 marks**]

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7. Write down the general equation for a geodesic curve in terms of an affine parameter λ.

The line element for a Schwarzschild spacetime is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 ,$$

where $c = 1$. Consider a photon moving in the equatorial plane ($\theta = \frac{\pi}{2}$) of the Schwarzschild spacetime whose coordinates are $x^\mu = (t, r, \theta, \phi)$. Show that

$$\frac{dr}{d\phi} = \frac{p^r}{p^\phi}$$

for the photon with four momentum $p^\mu = (p^0, p^r, p^\theta, p^\phi)$.

Use $p_\mu p^\mu = 0$ to show that the trajectory of the photon in this metric obeys

$$\left(\frac{p^r}{p^\phi}\right)^2 = r^4 \left(1 - \frac{2M}{r}\right) \left(\frac{\beta^2}{1 - \frac{2M}{r}} - \frac{1}{r^2}\right) ,$$

where

$$\beta = \frac{p_0}{p_\phi} .$$

Use the substitution $u = \frac{M}{r}$ to show

$$\left(\frac{du}{d\phi}\right)^2 = \beta^2 M^2 - u^2 + 2u^3 .$$

Differentiate the above equation to obtain

$$\frac{d^2u}{d\phi^2} + u = 3u^2 .$$

[20 marks]