PAPER CODE NO.
MATH326

THE UNIVERSITY
of LIVERPOOL

SUMMER 2004 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 4

RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\begin{aligned}
\Gamma^{\mu}_{\alpha\beta} &= \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}), \\
R^{\mu}{}_{\nu\sigma\rho} &= \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}, \\
R_{\mu\nu} &= R^{\sigma}{}_{\mu\sigma\nu}, \\
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \\
\gamma(v) &= \frac{1}{\sqrt{1 - v^2/c^2}} \\
c &= 2.998 \times 10^{8} \text{ ms}^{-1}.
\end{aligned}$$

1. State one of Einstein's principles of special relativity.

   An observer A is at rest at the origin of an inertial frame S. Observer B, in an inertial frame $S^{'}$, moves away from A with constant relative velocity v in the positive x direction. S and $S^{'}$ are synchronised at time $t = 0$. At time $t = T$ in S, A sends a light signal to B who receives it at $t^{'} = kT$ in $S^{'}$ and immediately reflects it back to A. Draw a spacetime diagram illustrating these events. Show that
$$k = \sqrt{\frac{c+v}{c-v}} \ .$$
   State the result if B moves *towards* A at constant relative velocity v.

   The Lorentz transformation between inertial frames S and $S^{'}$, where $S^{'}$ is moving at constant velocity v along the x-axis of S, is
$$t' = \gamma(v)\left(t - \frac{v}{c^2}x\right) \ , \quad x' = \gamma(v)(x - vt) \ .$$
   If $S^{''}$ is another inertial frame moving relative to $S^{'}$ at constant velocity u along the x'-axis, use the Lorentz transformation to show that the velocity w at which $S^{''}$ travels relative to S is given by
$$w = \frac{u+v}{1+\frac{uv}{c^2}} \ .$$
   Observer A is moving along the positive x-axis towards particle P at speed $4c/5$ in the rest frame of P. At some time, P emits a pair of particles Q and R. In P's rest frame Q moves along the x-axis in the positive direction with speed $2c/5$, whilst R moves in the negative direction along the x-axis with speed $2c/5$. What are the velocities of Q and R in the rest frame of A?

   If A emits a signal with frequency 1000Hz, and if Q is in the positive x direction as seen from A, what is the frequency as received by Q?

   [**20 marks**]

2. Define the momentarily comoving reference frame (MCRF).

The worldline of a particle in an inertial frame S is given by $x^\mu(\tau)$, where τ is the proper time. Define the 4-velocity U^μ and 4-acceleration a^μ of the particle.

Let the coordinates in S be $x^\mu = (ct, x, y, z)$ and let a particle be moving in the x-direction with velocity $v = dx/dt$. Show that

$$\frac{d(v\gamma(v))}{dt} = \gamma^3(v)\frac{dv}{dt} \ .$$

If $d(v\gamma(v))/dt = \alpha$, what is the name of the quantity denoted by α? What is its physical interpretation?

Suppose α is a constant and positive. If $x = 0$ and $v = 0$ at $t = 0$, show that the worldline of the particle moving along the positive x direction according to an observer at the origin of S is given by

$$x(t) = \frac{c^2}{\alpha}\left[\left(1 + \frac{\alpha^2 t^2}{c^2}\right)^{1/2} - 1\right] \ .$$

Show that $\gamma(v)$ for this worldline is given by

$$\gamma(v) = \left(1 + \frac{\alpha^2 t^2}{c^2}\right)^{1/2} \ .$$

Proper time satisfies $d\tau/dt = 1/\gamma(v)$. By differentiating, show that

$$t = \frac{c}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right)$$

is a solution of this equation for the above worldline.

When 17,000 years (1 year = 365 days) have elapsed in the inertial frame S, how many years and days (to the nearest day) have elapsed for an observer on the particle if $\alpha = 10\text{ms}^{-2}$?

[20 marks]

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3. A stationary particle of rest mass $6m$ at the origin is struck by a particle of rest mass m and total energy E travelling in the positive x direction. Two identical particles, each of rest mass $4m$, result from the collision, with one moving off at angle θ to the positive x direction and the other at angle $-\theta$, where $\theta = 0$ corresponds to motion along the positive x direction.

   Show that the two particles resulting from the collision must have the same energy, and that

$$\cos^2\theta = \frac{E^2 - m^2c^4}{E^2 + 12Emc^2 - 28m^2c^4} \ .$$

   Sketch $\cos^2\theta$ as a function of E for $E > 2mc^2$. Show that E must be at least $2.25mc^2$ and that the largest possible value for θ is 49.86°. To what incident energy E does this value of θ correspond?

   [20 marks]

4. The two dimensional plane is covered by a Cartesian set of coordinates $x^\mu = (x, y)$ and by plane polar coordinates $x^{\mu'} = (r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Compute

$$\Lambda^{\mu'}{}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}$$

   as a function of r and θ and hence show that

$$(\Lambda^{\mu}{}_{\mu'}) = \begin{pmatrix} \cos\theta & -r\ \sin\theta \\ \sin\theta & r\ \cos\theta \end{pmatrix} \ .$$

   Let V^μ, $T^\mu{}_\nu$ be respectively rank 1 and rank 2 tensors. Write down the transformation rule for each tensor between coordinate systems x^μ and $x^{\mu'}$ and the definition of their covariant derivatives with respect to x^σ.

   Let

$$V^\mu = \begin{pmatrix} y(y^2 - 3x^2) \\ x(3y^2 - x^2) \end{pmatrix} \ .$$

   Show that $V^x = -r^3 \sin 3\theta$ and $V^y = -r^3 \cos 3\theta$. By using the transformation rule show that $V^r = -r^3\ \sin 4\theta$ and $V^\theta = -r^2\ \cos 4\theta$.

   If the only non-zero Christoffel symbols for plane polar coordinates are $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$ show that

$$V^r{}_{;\theta} = -3r^3\ \cos 4\theta \qquad V^r{}_{;\theta r} = -6r^2\ \cos 4\theta \ .$$

   State, without calculation and with reasons, the value of $V^r{}_{;r\theta}$.

   [20 marks]

5. Consider a two-dimensional surface with line element given by

$$ds^2 = d\theta^2 + \sec^2\theta d\phi^2 \ .$$

Write down $g_{\mu\nu}$ and deduce its inverse $g^{\mu\nu}$. Compute the Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$ and show that the only non-zero components are given by

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \tan\theta \quad , \quad \Gamma^{\theta}_{\phi\phi} = -\tan\theta \ \sec^2\theta \ .$$

Hence using these expressions show that the value of the independent component of the Riemann tensor is given by $R^{\theta}{}_{\phi\theta\phi} = \sec^2\theta(1 - 2\sec^2\theta)$.

Using the symmetry properties of the Riemann tensor deduce the value of $R^{\phi}{}_{\theta\phi\theta}$. Hence show that the non-zero components of the Ricci tensor are given by $R_{\theta\theta} = (1 - 2\sec^2\theta)$ and $R_{\phi\phi} = \sec^2\theta(1 - 2\sec^2\theta)$ and compute the Ricci scalar. Calculate the Einstein tensor.

[**20 marks**]

6. The metric for a Schwarzschild spacetime is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \ ,$$

where $c = 1$ and M is constant. With respect to the coordinate system $x^\mu = (t, r, \theta, \phi)$ write down any quantities which are conserved on free particle trajectories, briefly giving reasons.

A photon moves in this spacetime along a worldline in the plane $\theta = \pi/2$. If $p_t = E$, $p_\theta = 0$ and $p_\phi = -L$ where E and L are constants, determine the particle momentum components p^t, p^θ and p^ϕ. With parameter λ defined by $p^r = dr/d\lambda$, show that r satisfies an equation of the form

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r) \equiv E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} \ .$$

Calculate the value of r corresponding to the zero of $V^2(r)$ and discuss the behaviour of $V^2(r)$ as $r \to \infty$. Determine the nature of the turning point of $V^2(r)$. Sketch the form of $V^2(r)$ and discuss the possible types of photon trajectory. If $M = 10/\sqrt{3}$ and $E = 10$, for what values of L is an unstable circular orbit possible?

[**20 marks**]

7. The equations governing the motion of a planet orbiting a star of mass M in the equatorial plane of a Schwarzschild spacetime with coordinates $x^\mu = (t, r, \theta, \phi)$ are

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 + \frac{\tilde{L}^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)$$

and

$$\frac{d\phi}{d\tau} = -\frac{\tilde{L}}{r^2}$$

where τ is the proper time, $\tilde{E}$ and $\tilde{L}$ are constants and $c = 1$. If $u = 1/r$ show that

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\left(\tilde{E}^2 - 1\right)}{\tilde{L}^2} + \frac{2Mu}{\tilde{L}^2} - u^2 + 2Mu^3 \ .$$

Setting $u = M/\tilde{L}^2 + y(\phi)$ and neglecting terms of order y^3 or higher, show that $y(\phi)$ satisfies an equation of the form

$$\left(\frac{dy}{d\phi}\right)^2 = a + by(\phi) + gy^2\left(\phi\right) \ .$$

Show that

$$a = \frac{\left(\tilde{E}^2 - 1\right)}{\tilde{L}^2} + \frac{M^2}{\tilde{L}^4} + \frac{2M^4}{\tilde{L}^6} \ ,$$

and find b and g in terms of $\tilde{L}$ and M.

Show that the function $y(\phi) = y_\text{o} + A\ \cos(k\phi)$ satisfies this equation, where y_o, A and k are constants, and determine the constants y_o and k in terms of M and $\tilde{L}$. Deduce from this solution that the perihelion advance of one orbit in the limit $M/\tilde{L} << 1$ is given by $6\pi M^2/\tilde{L}^2$.

For a nearly circular orbit $\tilde{L}^2 = MR$, where R is the radius of the orbit. Using this, calculate the ratio of the perihelion advance of Venus to the perihelion advance of Mercury after one orbit of Mercury, given that the radius of the orbits of Venus and Mercury are 108.2×10^6km and 55.4×10^6km respectively, and that Mercury orbits 2.6 times for each Venus orbit.

[**20 marks**]