MATH326 Summer 2001

Instructions to candidates.

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R = R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}$$

$$\gamma(v) = 1/\sqrt{(1 - v^2/c^2)}$$

1. State clearly one of Einstein's principles of special relativity.

An inertial frame S' moves at a constant speed u in the positive x direction relative to an inertial frame S. Write down the Lorentz transformation relating coordinates (ct', x') in S' to coordinates (ct, x) in S carefully defining any variables.

A third inertial frame S'' moves at constant speed v relative to S' along the positive x-axis. Its speed w relative to S is given by

$$w = \frac{u+v}{1+\frac{uv}{c^2}} .$$

If $\chi(u)$, $\chi(v)$ and $\chi(w)$ are the rapidities of the relative transformations show that

$$\chi(w) = \chi(u) + \chi(v) .$$

If $|u| < c$ and $|v| < c$ then deduce that $|w| < c$.

A particle is at rest at the origin O. At time $t = 0$ it decays into two identical particles which travel in opposite directions along the x-axis with constant speed $4c/9$. What is the speed of one particle relative to the other?

[20 marks]

2. Define the momentarily comoving reference frame, (MCRF).

A particle is moving with velocity $\mathbf{u} = (u_x, u_y, u_z)$ with respect to an observer in an inertial frame S. According to an observer in an inertial frame S' moving at constant relative velocity v in the positive x direction relative to S, the velocity of the same particle is $\mathbf{u}' = (u'_x, u'_y, u'_z)$ where

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} , \quad u'_y = \frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)} .$$

If the relation between the infinitesimal coordinates is

$$dt' = \gamma(v)\left(dt - \frac{v}{c^2}dx\right)$$

show that the x and y components of the particle's acceleration $\mathbf{a} = d\mathbf{u}/dt$ and $\mathbf{a}' = d\mathbf{u}'/dt'$ according to the respective observers, are related by

$$a'_x = \frac{a_x}{\gamma^3(v)\left(1 - \frac{u_x v}{c^2}\right)^3}$$

and

$$a'_y = \frac{1}{\gamma^2(v)\left(1 - \frac{u_x v}{c^2}\right)^2}\left(a_y + \frac{v u_y a_x}{c^2\left(1 - \frac{u_x v}{c^2}\right)}\right) .$$

If at some instant in S', $\mathbf{u}' = 0$ and $\mathbf{a}' = (a'_1, a'_2, a'_3)$ where a'_1, a'_2 and a'_3 are constants, show that $a_x = a'_1/\gamma^3(v)$ and $a_y = a'_2/\gamma^2(v)$. [20 marks]

3. A particle of rest mass $2m$ and energy E strikes a stationary particle of rest mass $5m$ to produce two identical particles of rest mass $4m$. They move off at an angle θ to the incident particle and on opposite sides. Show, by using conservation of energy-momentum, that

$$\cos^2\theta = \frac{(E^2 - 4m^2c^4)}{(E - 3mc^2)(E + 13mc^2)} .$$

Sketch the graph of $\cos^2\theta$ versus $x = E/(mc^2)$ in the region $x > 3$. Determine the maximum value of θ. What is the least value of E for the scattering to be possible?

[20 marks]

4. Define the covariant derivative of the vector V^μ in terms of its ordinary derivative and the Christoffel symbols. If x^μ and $x^{\mu'}$ are coordinates in two references frames, write down the relation between the components of the covariant derivative in the primed system in terms of those in the unprimed system and the transformation matrix Λ and its inverse.

Consider Cartesian coordinates $x^\mu = (x, y)$ on a flat two dimensional surface and plane polar coordinates $x^{\mu'} = (r, \theta)$. If they are related by

$$x = r\cos\theta \qquad y = r\sin\theta$$

show that the transformation matrix

$$\Lambda^{-1} = \frac{\partial x^\mu}{\partial x^{\mu'}} \equiv \left(\Lambda^\mu{}_{\mu'}\right)$$

is given by

$$\Lambda^{-1} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} .$$

Hence deduce Λ.

A vector V^μ has Cartesian coordinates $(-y, x)$. Compute $V^\mu{}_{,\nu}$ and show that

$$\Lambda^{\mu'}{}_\mu V^\mu{}_{,\nu} \Lambda^\nu{}_{\nu'} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix} .$$

Hence write down all the components of $V^{\mu'}{}_{;\nu'}$ clearly justifying your reasoning.

[20 marks]

5. The line element for a two dimensional surface is

$$ds^2 = y^m dx^2 + x^n dy^2$$

where m and n are non-negative integers. Write down the components of the metric tensor and its inverse. Show that

$$\Gamma^x_{\ xy} = \frac{m}{2y} \quad \text{and} \quad \Gamma^y_{\ xx} = -\frac{my^{m-1}}{2x^n}$$

and compute the remaining components of the Christoffel symbol.

Show that for the given line element

$$R^y{}_{xyx} = -\frac{n(n-2)}{4x^2} - \frac{m(m-2)y^{m-2}}{4x^n} .$$

Hence, by using the metric tensor, its inverse and the symmetries of the Riemann tensor show that

$$R^x{}_{yxy} = -\frac{m(m-2)}{4y^2} - \frac{n(n-2)x^{n-2}}{4y^m} .$$

Using the expressions for $R^x{}_{yxy}$ and $R^y{}_{xyx}$, calculate the three independent components of the Ricci tensor and show it is proportional to the metric tensor.

[20 marks]

6. The metric for a Schwarzschild spacetime is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

where $c = 1$ and M is constant in the coordinate system $x^\mu = (t, r, \theta, \phi)$. Write down the components of $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

A particle of non-zero mass m freely moves along a worldline $x^\mu(\tau)$ in the equatorial plane $\theta = \pi/2$ of this spacetime. If the components of its momentum, p_μ, are given by

$$p_t = m\tilde{E} \ , \ p_\theta = 0 \ \text{and} \ p_\phi = -m\tilde{L}$$

where $\tilde{E}$ and $\tilde{L}$ are constants, compute p^t, p^θ and p^ϕ.

Hence find the effective potential $\tilde{V}^2(r)$ of the radial motion given by

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r) .$$

If $\tilde{L} = 10\sqrt{3}$ and $M = 3$ show that for the stable circular orbit

$$\tilde{E} = \frac{14}{9}\sqrt{\frac{2}{5}} .$$

[20 marks]

7. Consider a photon moving in the equatorial plane of the Schwarzschild spacetime whose coordinates are $x^\mu = (t, r, \theta, \phi)$ where $c = 1$. The trajectory is governed by

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r) \quad \text{and} \quad \frac{d\phi}{d\lambda} = \frac{L}{r^2}$$

where M is the mass of the gravitational source, λ parametrises the worldline of the photon and

$$V^2(r) = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) .$$

Plot $V^2(r)$ in the region $r > 2M$, clearly labelling all stationary points and showing the asymptotic behaviour. Describe the nature of all possible trajectories.

For an incoming photon with $\tilde{L} > 0$ and $d\phi/dr < 0$, show that

$$\frac{d\phi}{du} = \left(\frac{1}{b^2} - u^2(1 - 2Mu)\right)^{-1/2}$$

where $r = 1/u$ and $b = L/E$. Solve the equation in the case when $M = 0$ and describe the geometry of the trajectory.

For $M \neq 0$, show that ϕ satisfies

$$\frac{d\phi}{dy} = (1 + 2My)\left(\frac{1}{b^2} - y^2\right)^{-1/2}$$

when $y = u(1 - Mu)$, which implies $u = y(1 + My)$, and terms of order y^3 and higher are neglected relative to y^2 in the differential equation.

Show, by differentiating the right side of

$$\phi = \phi_0 + \sin^{-1}(by) + \frac{2M}{b} - 2M\left(\frac{1}{b^2} - y^2\right)^{1/2}$$

that it satisfies the differential equation for $\phi(y)$.

[20 marks]