MATH326 Summer 2000

Instructions to candidates.

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R = R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}$$

$$\gamma(v) = 1/\sqrt{(1 - v^2/c^2)}$$

1. The Lorentz transformation between inertial frames S and S', where S' moves at constant speed v in the positive direction along the x-axis of S away from the origin, is

$$t' = \gamma(v)\left(t - \frac{v}{c^2}x\right) \qquad x' = \gamma(v)\,(x - vt) \quad .$$

If S'' is another inertial frame moving relative to S' at constant speed u along the postive x'-axis, determine the speed which S'' travels at relative to S.

Find $k(v)$ where

$$t' - \frac{x'}{c} = k(v)\left(t - \frac{x}{c}\right)$$

and compute $(t'^2 - x'^2/c^2)$ in terms of $(t^2 - x^2/c^2)$. State, giving reasons, the value of $(t''^2 - x''^2/c^2)$.

If the frequency of a light signal emitted in S is f and it has an observed frequency of f' in S', show that

$$f' = \frac{f}{k(v)} \quad .$$

A star moves at speed $0{\cdot}9c$ away from the earth. To one decimal place, what frequency does an observer on the star measure for a light signal whose frequency as measured on earth is 1000 Hertz? If a second star moves at speed $0{\cdot}8c$ away from the first star along the same direction, what is the frequency of the light signal as seen by an observer on the second star?

2. The worldline of a particle in some inertial frame S is given by $x^\mu(\tau)$ where τ is the proper time. Define the 4-velocity U^μ and 4-acceleration a^μ and state the value of U^μ in the momentarily comoving reference frame. If the coordinates in S are $x^\mu = (ct, x, y, z)$ and $v = dx/dt$, show that

$$\frac{d}{dt}\left(v\gamma(v)\right) = \gamma^3(v)\frac{dv}{dt} \quad .$$

If the worldline of the particle according to an observer at the origin O of S is

$$x(t) = \frac{c^2}{a}\left[\left(1 + \frac{a^2t^2}{c^2}\right)^{1/2} - 1\right]$$

where a is constant, compute $d(v\gamma(v))/dt$. What is the value of the particle's proper acceleration?

If $d\tau/dt = 1/\gamma(v)$, show that for this worldline

$$\tau = \frac{c}{a}\sinh^{-1}\left(\frac{at}{c}\right)$$

where clocks are synchronised at the origin at $t = \tau = 0$. When the proper time of the particle is 365 days, what time has elapsed for the observer in S if $a = 10\ \mathrm{ms}^{-2}$?

3. A particle of rest mass m_1 and energy E strikes a stationary particle of rest mass m_2 to produce two identical particles of rest mass m_3. They move off at an angle θ to the direction of the incident particle and on opposite sides. Show, by using conservation of energy-momentum, that

$$\cos^2\theta \;=\; \frac{(E^2 - m_1^2c^4)}{(E + m_2c^2 - 2m_3c^2)(E + m_2c^2 + 2m_3c^2)} \;.$$

If $m_1 = 2m$, $m_2 = 3m$ and $m_3 = 3m$ sketch the graph of $\cos^2\theta$ versus $x = E/(mc^2)$ in the region $x > 3$. By considering the minimum value of $\cos^2\theta$ show that θ must be less than $33{\cdot}02^{\text{o}}$. What is the least value of E for the scattering to be possible?

4. The two dimensional plane is covered by a Cartesian set of coordinates $x^\mu = (x, y)$ and by plane polar coordinates $x^{\mu'} = (r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Compute

$$\Lambda^{\mu'}{}_{\mu} \;=\; \frac{\partial x^{\mu'}}{\partial x^\mu}$$

as a function of r and θ and show that

$$\left(\Lambda^{\mu}{}_{\mu'}\right) \;=\; \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \;.$$

A tensor T has components

$$T^{\mu}{}_{\nu} \;=\; \begin{pmatrix} x^2 - y^2 & 2xy \\ -2xy & x^2 - y^2 \end{pmatrix} \;.$$

Show that the components of $T^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_{\mu} T^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\nu'}$ are

$$T^{\mu'}{}_{\nu'} \;=\; \begin{pmatrix} r^2\cos 2\theta & r^3\sin 2\theta \\ -r\sin 2\theta & r^2\cos 2\theta \end{pmatrix} \;.$$

If the only non-zero components of the metric connection for plane polar coordinates are

$$\Gamma^r_{\theta\theta} \;=\; -\,r \quad , \quad \Gamma^\theta_{\theta r} \;=\; \Gamma^\theta_{r\theta} \;=\; \frac{1}{r} \;,$$

show that $T^r{}_{r;r} = 2r\cos 2\theta$ and $T^r{}_{\theta;\theta} = 2r^3\cos 2\theta$. State with clear reasoning the value of $T_r{}^\theta{}_{;\theta}$.

[You may quote the formula $T^\mu{}_{\nu;\sigma} \;=\; T^\mu{}_{\nu,\sigma} + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu \;-\; \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho$.
The line element for plane polar coordinates is $ds^2 = dr^2 + r^2 d\theta^2$.]

5. The line element for a two dimensional surface is

$$ds^2 \;=\; d\theta^2 \;+\; \sin^n\theta d\phi^2$$

where n is constant. Write down the components of $g^{\mu\nu}$. Compute the components of the metric connection $\Gamma^\mu_{\nu\sigma}$ and show that the only non-zero components are

$$\Gamma^\phi_{\theta\phi} \;=\; \Gamma^\phi_{\phi\theta} \;=\; \frac{n}{2}\cot\theta \;\;,\;\; \Gamma^\theta_{\phi\phi} \;=\; -\,\frac{n}{2}\sin^{n-1}\theta\cos\theta \;\;.$$

With these values show that

$$R^\theta{}_{\phi\theta\phi} \;=\; \frac{1}{4}n^2\sin^n\theta \;-\; \frac{1}{4}n(n-2)\sin^{n-2}\theta$$

and hence deduce the value of $R^\phi{}_{\theta\phi\theta}$, clearly showing your reasoning.

Show that the Ricci tensor is given by

$$R_{\mu\nu} \;=\; \frac{n}{4}[n \;-\; (n-2)\mathrm{cosec}^2\theta]g_{\mu\nu}$$

and deduce R. For which values of n is the curvature of the surface constant?

6. The metric for a Schwarzschild spacetime is given by

$$(ds)^2 \;=\; \left(1-\frac{2M}{r}\right)c^2(dt)^2 \;-\; \left(1-\frac{2M}{r}\right)^{-1}(dr)^2 \;-\; r^2(d\theta)^2 \;-\; r^2\sin^2\theta\,(d\phi)^2$$

where $c = 1$ and M is constant. With respect to the coordinate system $x^\mu = (t, r, \theta, \phi)$ write down any quantities which are conserved on free particle trajectories, briefly giving reasons.

A particle of mass m freely moves in this spacetime along a worldline $x^\mu(\tau)$ in the plane $\theta = \pi/2$. If $p_t = m\tilde{E}$, $p_\theta = 0$ and $p_\phi = -\,m\tilde{L}$ where p^μ is the particle momentum and $\tilde{E}$ and $\tilde{L}$ are constants, determine p^t, p^θ and p^ϕ. Show that $dr/d\tau$ satisfies an equation of the form

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; \tilde{E}^2 \;-\; V^2(r)$$

and find $V^2(r)$.

If $\tilde{L} = 5$ and $M = \sqrt{3}/2$ sketch the function $V^2(r)$ in the region $r > 2M$ and show that for a stable circular orbit

$$\tilde{E} \;=\; \frac{14\sqrt{10}}{45} \;\;.$$

To one decimal place, what is the least radius that a particle in an elliptical orbit can have without capture by the gravitational source?

7. The function $y(\phi)$ is given by $y(\phi) = y_0 + A\cos(k\phi)$ where y_0, A and k are constants. Show that $y(\phi)$ satisfies a differential equation of the form

$$\left(\frac{dy}{d\phi}\right)^2 = \alpha + \beta y(\phi) + \gamma y^2(\phi)$$

and deduce α, β and γ in terms of y_0, A and k.

The equations governing the motion of a planet of mass M orbiting in the equatorial plane of a Schwarzschild spacetime are

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 + \frac{\tilde{L}^2}{r^2}\right)\left(1 - \frac{2M}{r}\right) \quad \text{and} \quad \frac{d\phi}{d\tau} = -\frac{\tilde{L}}{r^2}$$

where $\tilde{E}$ and $\tilde{L}$ are constants and $c = 1$. If $u = 1/r$ show that

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(\tilde{E}^2 - 1)}{\tilde{L}^2} + \frac{2Mu}{\tilde{L}^2} - u^2 + 2Mu^3 .$$

Setting $u = M/\tilde{L}^2 + y(\phi)$ and neglecting terms of order y^3 and higher, show that $y(\phi)$ satisfies an equation of the form

$$\left(\frac{dy}{d\phi}\right)^2 = \bar{\alpha} + \bar{\beta} y(\phi) + \bar{\gamma} y^2(\phi)$$

and deduce $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ in terms of $\tilde{E}$, $\tilde{L}$ and M. Hence determine the constants y_0 and k which give a solution of the form $y(\phi) = y_0 + A\cos(k\phi)$.

If $M/\tilde{L} << 1$ show that the perihelion advance of one orbit is approximately $6\pi M^2/\tilde{L}^2$.