PAPER CODE NO.
MATH325

THE UNIVERSITY
of LIVERPOOL

JANUARY 2005 EXAMINATIONS

Bachelor of Arts	:	Year 3
Bachelor of Science	:	Year 2
Bachelor of Science	:	Year 3
Master of Mathematics	:	Year 3
Master of Mathematics	:	Year 4

QUANTUM MECHANICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

In this paper, bold-face quantities such as **r** represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted. Marks for parts of questions may be subject to small adjustments

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1. (i) A three-state system has the Hamiltonian

$$\hat{H} = \begin{pmatrix} U & 0 & 0 \\ 0 & 2U & ia \\ 0 & -ia & 2U \end{pmatrix}$$

where U and a are real constants. Check that the state

$$|\psi\rangle = n \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

is an eigenstate of $\hat{H}$. What is its eigenvalue? Determine the normalisation constant n, which you should choose to be real and positive.

An observable Q is represented by the matrix

$$\hat{Q} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

What is the average value which will be found for Q if it is measured for a system in the state $|\psi\rangle$?

(ii) A particle at some moment in time is described by the wave function

$$\psi(x) = \begin{cases} A(a - |x|) & : \quad |x| \leq a \\ 0 & : \quad \text{otherwise}, \end{cases}$$

where A and a are real positive constants. Find the normalisation constant A in terms of a.

Find the expectation values $\langle \hat{p} \rangle$ and $\langle \hat{p}^2 \rangle$ with respect to the given wave function.

Deduce that the uncertainty Δp in a measurement of the momentum of the particle is given by

$$\Delta p = \frac{\hbar}{a} \sqrt{3} .$$

[20 marks]

2. A particle of mass m is confined to the region $0 \leq x \leq L$ of the x-axis. Inside this region the potential energy is 0, outside this region it is infinite. Write down the corresponding time-independent Schrödinger equation for the problem and hence find the normalised eigenfunctions of the Hamiltonian.

   Show that the energy eigenvalues are

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad (n = 1, 2, 3 \ldots)$$

   At a particular moment, the particle is in a state described by the normalised wave function

$$\psi(x) = \begin{cases} Cx(L-x) & : \quad 0 \leq x \leq L \\ 0 & : \quad x < 0 \quad \text{or} \quad x > L \end{cases}$$

   where C is a real, positive normalisation constant.

   (i) Determine the normalisation constant C.

   (ii) Calculate the expectation value of the energy.

   (iii) Calculate the probability, expressed as a percentage, that a measurement of the energy will give the result E_1.

[20 marks]

3. A beam of particles of mass m and energy E is incident in the positive x direction on a potential barrier whose potential V is given by

$$V(x) = \begin{cases} 0 & : \; x < 0 \qquad \text{(region I)} \\ U & : \; 0 \le x \le L \quad \text{(region II)} \\ 0 & : \; x > L \qquad \text{(region III)} \end{cases}$$

where $0 < E < U$.

(a) Show that the particle wave function in the x regions defined above can be written

$$\begin{aligned} \psi_{\text{I}}(x) &= Ae^{iqx} + Be^{-iqx} \\ \psi_{\text{II}}(x) &= Ce^{Kx} + De^{-Kx} \\ \psi_{\text{III}}(x) &= Fe^{iqx} \end{aligned}$$

where you should find expressions for K and q.

(b) State the conditions on the particle wave function ψ and its derivative ψ' which must be satisfied at the boundaries between regions I, II and III and use these to show that

$$\frac{A}{F} = \frac{e^{iqL}}{4i\,qK} \left\{ (K + iq)^2 \, e^{-KL} - (K - iq)^2 \, e^{KL} \right\}$$

The incident particle flux for the above scattering problem is defined by

$$j_I = \frac{\hbar q}{m} |A|^2 \, .$$

Give the corresponding expressions for the reflected and transmitted fluxes j_R and j_T. Hence define the reflection and transmission coefficients R and T.

(c) Use the result of part (b) to evaluate the transmission coefficient T.

[20 marks]

4. The Hamiltonian for a particle of mass m moving on the x-axis in a harmonic oscillator potential can be written in the form

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega$$

where the frequency ω is a positive constant, and where $[a, a^\dagger] = 1$. The position x and momentum p are given by

$$x = \frac{i}{\alpha\sqrt{2}}(a - a^\dagger) \quad \text{and} \quad p = \frac{\hbar\alpha}{\sqrt{2}}(a + a^\dagger),$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

There is a normalised state $|\psi_0\rangle$ which satisfies

$$a|\psi_0\rangle = 0.$$

(i) Show that $|\psi_0\rangle$ is an eigenstate of H, and find the eigenvalue.

(ii) Using the commutation relations prove that

$$|\psi_1\rangle = C_1 a^\dagger |\psi_0\rangle$$

is also an eigenstate of energy, and find the eigenvalue. Determine the value which the constant C_1 must take if $|\psi_1\rangle$ is properly normalised (you can choose C_1 to be real and positive).

(iii) By induction (or otherwise) prove that

$$|\psi_n\rangle = C_n \left(a^\dagger\right)^n |\psi_0\rangle$$

is an eigenstate of H. Find the eigenvalue, and find the normalisation constant C_n.

(iv) Show that

$$a|\psi_n\rangle = \sqrt{n}\, |\psi_{n-1}\rangle \ .$$

(v) By writing $x|\psi_n\rangle$ in terms of $|\psi_{n-1}\rangle$ and $|\psi_{n+1}\rangle$, compute $\langle x^2 \rangle$ for the state $|\psi_n\rangle$.

[20 marks]

5. Angular momentum is represented by operators L_i $(i = 1, 2, 3)$ which satisfy the commutation relations $[L_1, L_2] = i\hbar L_3$ (and cyclic permutations). Use these commutation relations to show that

$$[\mathbf{L}^2, L_3] = 0$$

where $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$.

For this question you can assume the following properties of the angular momentum operators and their eigenstates. There are normalised states $|l, m\rangle$ which are simultaneously eigenstates of $\mathbf{L}^2$ and L_3, satisfying

$$\begin{aligned} L_3|l, m\rangle &= \hbar m|l, m\rangle \\ \mathbf{L}^2|l, m\rangle &= \hbar^2 l(l+1)|l, m\rangle \end{aligned} \qquad (1)$$

where $2l$ is a positive integer and $m \in \{-l, -l+1, \cdots l-1, l\}$. Moreover there are raising and lowering operators $L_+ = L_1 + iL_2$ and $L_- = L_1 - iL_2$ which act on the eigenstates according to

$$L_+|l, m\rangle = M_{l,m}|l, m+1\rangle \quad \text{and} \quad L_-|l, m\rangle = N_{l,m}|l, m-1\rangle, \qquad (2)$$

with

$$M_{l,m} = \hbar\sqrt{l(l+1) - m(m+1)} \quad \text{and} \quad N_{l,m} = \hbar\sqrt{l(l+1) - m(m-1)}\,.$$

(i) Use the commutation relations to show that

$$L_+L_- = \mathbf{L}^2 - L_3^2 + \hbar L_3\,. \qquad (3)$$

(ii) A particle is in a state such that $l = \frac{3}{2}$. Write down the allowed values of m (corresponding to the eigenvalues of L_3) and use equations (1) and (2) to write down 4×4 matrices representing L_3, L_+ and L_-.

(iii) Check whether your matrices satisfy equation (3).

(iv) Use your matrix representations for L_+, L_- to find matrix representations for L_1 and L_2.

[20 marks]

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6. (i) Using integration by parts, or otherwise, show that for $n > 0$

$$I_n = \frac{n}{\alpha} I_{n-1}\,, \qquad \text{where} \quad I_n \equiv \int_0^\infty r^n e^{-\alpha r}\, dr\,.$$

Evaluate I_0 and deduce the value of I_3.

(ii) The Hamiltonian for a particle of mass m moving in three dimensions under the influence of an attractive Coulomb potential is

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{K}{r}\,,$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and the radial part of the Laplacian operator is

$$\nabla^2_{\text{rad}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

Given that the normalised ground state wave function is

$$\psi_0(\mathbf{r}) = A e^{-\frac{1}{2}\alpha r}\,,$$

where A is real, determine α and the ground state energy E_0. Also find the normalisation constant A.

(iii) The potential is perturbed by the addition of a term σr where σ is a small constant. Use first order perturbation theory to obtain an approximation to the perturbed ground state energy in the form

$$E_0 + \sigma C$$

where C is a constant which you should find.

[*Standard results from perturbation theory may be assumed without proof.*]

[20 marks]

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7. (i) Give a statement of the variational principle and explain briefly how it may be used to obtain an upper bound on the ground state energy E_0 of a system with Hamiltonian $\hat{H}$.

(ii) The motion of a particle of mass m in one dimension is described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + Kx^4 \qquad (K > 0)\,.$$

Consider the following trial wave function

$$\psi(x) = Ae^{-\alpha|x|}$$

where $\alpha > 0$. Calculate the normalisation constant A. Use the variational method to find an upper bound for the ground state energy.

Note: in (ii) you may use without proof the result

$$I_n(b) \equiv \int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

when $b > 0$.

[20 marks]