PAPER CODE NO.
MATH325

THE UNIVERSITY
of LIVERPOOL

JANUARY 2003 EXAMINATIONS

Bachelor of Arts	:	Year 3
Bachelor of Science	:	Year 2
Bachelor of Science	:	Year 3
Master of Mathematics	:	Year 3
Master of Mathematics	:	Year 4

QUANTUM MECHANICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

In this paper, bold-face quantities such as $\mathbf{r}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted. Marks for parts of questions may be subject to small adjustments

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1. (i) Determine which, if any, of the following operators could represent an observable in quantum mechanics

$$\hat{A} = x\frac{d}{dx}\,, \quad \hat{B} = i(x\frac{d}{dx} + \frac{1}{2})$$

stating clearly any assumptions you make.

[Hint: you may use the property of a Hermitian operator that

$$\langle \hat{A}\psi|\phi\rangle = \langle \psi|\hat{A}\phi\rangle$$

where $\psi(x)$ and $\phi(x)$ are any two normalisable wave functions]

(ii) A particle at some moment in time is described by the wave function

$$\psi(x) = \begin{cases} c(a^2 - x^2) & : \quad |x| \le a \\ 0 & : \quad \text{otherwise}\,, \end{cases}$$

where c and a are real positive constants. Find the normalisation constant c in terms of a.

Find the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{x}^2 \rangle$ with respect to the given wave function.

Deduce that the uncertainty Δx in a measurement of the position of the particle is given by

$$\Delta x = \frac{a}{\sqrt{7}}\,.$$

[20 marks]

2. A particle of mass m is confined to the region $0 \leq x \leq L$ of the x-axis. Write down the corresponding time-independent Schrödinger equation for the problem and hence find the normalised eigenfunctions of the Hamiltonian.

Show that the energy eigenvalues are

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad (n = 1, 2, 3 \ldots)$$

At a particular moment, the particle is in a state described by the normalised wave function

$$\psi(x) = \begin{cases} -Ax & : \quad 0 \leq x \leq \frac{L}{2} \\ A(x - L) & : \quad \frac{L}{2} < x \leq L \\ 0 & : \quad x < 0 \quad \text{or} \quad x > L \end{cases}$$

where A is a real, positive normalisation constant.

(i) Determine the normalisation constant A.

(ii) Calculate the probability, expressed as a percentage, that a measurement of the energy will give the result E_1.

[20 marks]

3. A beam of identical particles of mass m and energy $E > 0$ is travelling along the x-axis from $x < 0$ and is incident on a potential step

$$\begin{aligned} V(x) &= V_0 \qquad x \geq 0 \\ V(x) &= 0 \qquad x < 0 \end{aligned}$$

where V_0 is a constant. Suppose that $0 < E < V_0$.

(i) Write down an expression for the current density j_I for a beam of particles with wave-function $\psi(x) = Ae^{ikx}$. For the potential step above, evaluate the reflection coefficient R, defined as the ratio of the reflected current density to the incident current density.

(ii) Deduce the transmission coefficient T, and comment on the result.

(iii) Calculate the relative probability of finding a particle at position x (> 0) compared with that of finding one at the origin $(x = 0)$. Comment on the physical significance of this result.

(iv) Consider, instead, the case $E > V_0$. Describe, without further calculation, in what respect you would expect the nature of the solution to differ from that which you have already provided.

[20 marks]

4. The Hamiltonian for a particle of mass m moving on the x-axis in a harmonic oscillator potential can be written in the form

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega$$

where the frequency ω is a positive constant, and where $[a, a^\dagger] = 1$. The position x and momentum p are given by

$$x = \frac{i}{\sqrt{2}\alpha}(a - a^\dagger) \quad \text{and} \quad p = \frac{\hbar\alpha}{\sqrt{2}}(a + a^\dagger),$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$.

(i) Show by induction that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$, for n a positive integer.

(ii) The normalised eigenfunctions of the Hamiltonian are given by

$$\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n \psi_0, \quad n \geq 0,$$

where $a\psi_0 = 0$. Show that

$$a\psi_n = \sqrt{n}\psi_{n-1} \qquad \text{and} \qquad a^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}.$$

(iii) By writing $x\psi_n$, $p\psi_n$ in terms of ψ_{n-1}, ψ_{n+1}, compute the uncertainties Δx and Δp for the state ψ_n.

(iv) Find $\Delta x \Delta p$ for the state ψ_n and comment on the result.

[You may find the following identity useful:

$$[A, BC] = B[A, C] + [A, B]C$$

for operators A, B and C.]

[20 marks]

5. Given that the angular momentum operators L_i $(i = 1, 2, 3)$ satisfy the commutation relations $[L_1, L_2] = i\hbar L_3$ (and cyclic permutations), show that
$$[\mathbf{L}^2, L_1] = [\mathbf{L}^2, L_2] = [\mathbf{L}^2, L_3] = 0$$
where $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$.

From the above commutation relations it is possible to deduce the following results (*which you may assume*). There exist normalised eigenstates $|l, m\rangle$ such that
$$L_3|l, m\rangle = \hbar m|l, m\rangle, \qquad \mathbf{L}^2|l, m\rangle = \hbar^2 l(l + 1)|l, m\rangle,$$
where $2l$ is a positive integer and the possible values of m are $-l, -l + 1, \ldots l - 1, l$. Moreover,
$$L_+|l, m\rangle = M_{l,m}|l, m + 1\rangle \quad \text{and} \quad L_-|l, m\rangle = N_{l,m}|l, m - 1\rangle,$$
where $L_+ = L_1 + iL_2$ and $L_- = L_1 - iL_2$, and $M_{l,m}$ and $N_{l,m}$ are real, positive constants.

(i) Show that
$$L_-L_+ = \mathbf{L}^2 - L_3^2 - \hbar L_3$$
and, by considering the norm of $L_+|l, m\rangle$, show that
$$M_{l,m} = \hbar\sqrt{l(l + 1) - m(m + 1)}\,.$$

(ii) A particle is in a state such that $l = 1$. Write down the allowed values of m (corresponding to the eigenvalues of L_3) and evaluate the matrix elements
$$\langle 1, 0|L_+|1, 0\rangle \qquad \text{and} \qquad \langle 1, 1|L_+|1, 0\rangle\,.$$

(iii) Find all other non-zero elements of the matrix
$$\langle 1, m'|L_+|1, m\rangle\,.$$
and display your results for the full 3×3 matrix where the rows are labelled by values of m' and the columns by values of m.

(iv) Obtain a similar matrix representation for L_-, and hence find a matrix representation for L_1.
[*You may assume that in (iv), $N_{l,m}$ is given by*
$$N_{l,m} = \hbar\sqrt{l(l + 1) - m(m - 1)}\,.]$$

[20 marks]

6. (i) Using integration by parts, or otherwise, show that for $n \geq 2$

$$I_n = \frac{n-1}{2\beta^2} I_{n-2}\,, \qquad \text{where} \quad I_n \equiv \int_0^\infty r^n e^{-\beta^2 r^2}\, dr\,.$$

Given that

$$I_0 = \frac{\sqrt{\pi}}{2\beta}\,,$$

find I_2. Evaluate I_1 and deduce the value of I_5.

(ii) The Hamiltonian for a particle of mass m moving in three dimensions under the influence of a three-dimensional harmonic oscillator potential is

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2} m\omega^2 r^2\,,$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and the radial part of the Laplacian operator is

$$\nabla^2_{\text{rad}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

Given that the normalised ground state wave function is

$$\psi_0(\mathbf{r}) = A e^{-\frac{1}{2}\beta^2 r^2}\,,$$

where A is real, determine β and the ground state energy E_0. Calculate also the normalisation constant A.

(iii) The potential is perturbed by the addition of a term λr^5 where λ is small. Use first order perturbation theory to obtain an approximation to the perturbed ground state energy in the form

$$E_0 + \lambda K$$

where K is a constant which you should find.

[*Standard results from perturbation theory may be assumed without proof.*]

[20 marks]

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7. (i) Give a statement of the variational principle and explain briefly how it may be used to obtain an upper bound on the ground state energy E_0 of a system with Hamiltonian $\hat{H}$.

   (ii) The motion of a particle of mass m in one dimension is described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \lambda|x| \qquad (\lambda > 0)\,.$$

   Consider, in turn, each of the following two normalised wave functions

$$\psi_1(x) = A_1 e^{-\alpha|x|} \qquad \text{and} \qquad \psi_2(x) = A_2(1+\alpha|x|)e^{-\alpha|x|}\,,$$

   Here, $\alpha > 0$ and

$$A_1 = \sqrt{\alpha}\,, \qquad A_2 = \sqrt{\frac{2\alpha}{5}}\,.$$

   By applying the variational method, decide which (if any) of these is a suitable trial wave function for the given problem. Where appropriate, give a variational upper bound for the ground state energy.

Note: in (ii) you may use without proof the result

$$I_n(b) \equiv \int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

when $b > 0$.

[20 marks]