PAPER CODE NO.
MATH325

THE UNIVERSITY
of LIVERPOOL

JANUARY 2002 EXAMINATIONS

Bachelor of Arts	:	Year 3
Bachelor of Science	:	Year 2
Bachelor of Science	:	Year 3
Master of Mathematics	:	Year 3
Master of Mathematics	:	Year 4

QUANTUM MECHANICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

In this paper, bold-face quantities such as **r** represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted. Marks for parts of questions may be subject to small adjustments

1. (i) A linear operator $\hat{A}$ acting on a space of normalisable wave functions $\{\psi(x)\}$ is said to be Hermitian if

$$\langle \hat{A}\psi|\phi\rangle = \langle\psi|\hat{A}\phi\rangle$$

for any two wave functions $\psi(x)$ and $\phi(x)$ and where $\langle\psi|\phi\rangle$ is the usual inner product.

Show that the 1-dimensional momentum operator

$$\hat{p}_x = -i\hbar\frac{d}{dx}$$

is Hermitian, stating clearly any assumptions that you make.

(ii) The wave function of a particle at some moment in time is given by

$$\phi(x) = e^{-\alpha^2x^2/2}\,, \qquad -\infty < x < \infty\,.$$

Find a corresponding normalised wave function $\phi_N(x)$.

Find the expectation values $\langle\hat{p}\rangle$ and $\langle\hat{p}^2\rangle$ with respect to the given wave function.

Deduce that the uncertainty Δp in a measurement of momentum is given by

$$(\Delta p)^2 = \frac{1}{2}\alpha^2\hbar^2\,.$$

If the momentum is measured to be $\alpha\hbar$, what is the probability that a *second* measurement of momentum, made immediately afterwards, would yield one half of this value?

You may use the results:

$$\int_{-\infty}^{\infty} e^{-\beta^2x^2}\,dx = \frac{\sqrt{\pi}}{\beta}\,, \qquad \int_{-\infty}^{\infty} x^2e^{-\beta^2x^2}\,dx = \frac{\sqrt{\pi}}{2\beta^3}\,.$$

[**20 marks**]

2. A particle of mass m is confined to the region of the x-axis between $x = 0$ and $x = L$. Write down an expression for the appropriate time-independent Hamiltonian.

   (i) Given that the corresponding normalised eigenfunctions are

   $$\phi_n(x) = A\sin\left(\frac{n\pi x}{L}\right) \qquad (n = 1, 2, 3\ldots),$$

   verify that the energy eigenvalues are E_n where

   $$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2} \qquad (n = 1, 2, 3\ldots)$$

   and determine the normalisation constant A.

   (ii) At a certain time, the particle is in a state described by the normalised wave function

   $$\psi(x) = \begin{cases} \lambda x & : \quad 0 \le x \le \frac{L}{2} \\ \lambda(L - x) & : \quad \frac{L}{2} < x \le L \\ 0 & : \quad x < 0 \quad \text{or} \quad x > L \end{cases}$$

   where λ is a real, positive normalisation constant.

   Determine the normalisation constant λ.

   Show that

   $$\langle\psi|\phi_1\rangle = \frac{4\sqrt{6}}{\pi^2}$$

   and deduce the probability that a measurement of the energy will give the result E_1. Express your answer as a percentage given to the nearest whole percent.

[20 marks]

3. A particle of mass m and energy E moves on the x-axis subject to a potential V given by

$$V(x) = \begin{cases} 0 & : \quad |x| > a \qquad \text{(regions I and III)} \\ -V_0 & : \quad |x| \leq a \qquad \text{(region II)} \end{cases}$$

where $E < 0$ and $V_0 > 0$. Suppose that $E > -V_0$ and define

$$q^2 = -\frac{2mE}{\hbar^2} \qquad \text{and} \qquad k^2 = \frac{2m(E + V_0)}{\hbar^2} \,.$$

(i) Write down the energy eigenfunction equation (time-independent Schrödinger equation) in the regions I, III and II. Hence show that energy eigenfunctions are either odd or even functions of x, giving explicit expressions for the eigenfunctions valid for all x.

(ii) Show that, for an odd solution, k must satisfy

$$k\cot(ka) = -\sqrt{\alpha^2 - k^2}$$

where

$$\alpha^2 = \frac{2mV_0}{\hbar^2} \,.$$

[20 marks]

4. The Hamiltonian of a particle of mass m undergoing simple harmonic motion along the x-axis is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

where

$$\hat{p} = -i\hbar\frac{d}{dx}$$

and ω is a positive constant.

(i) Show that, if we define

$$a = \frac{\alpha}{\sqrt{2}}(\frac{1}{m\omega}\hat{p} - i\hat{x}) \qquad \text{and} \qquad a^\dagger = \frac{\alpha}{\sqrt{2}}(\frac{1}{m\omega}\hat{p} + i\hat{x})$$

where $\alpha^2 = m\omega/\hbar$, then it follows from the commutator $[\hat{x},\hat{p}] = i\hbar$ that $[a, a^\dagger] = 1$.

(ii) Show by induction that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$.

(iii) Given that the normalised eigenfunctions of the Hamiltonian are

$$\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0 \qquad \text{where} \qquad a\psi_0 = 0\,,$$

show that

$$a\psi_n = \sqrt{n}\psi_{n-1} \qquad \text{and} \qquad a^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}\,.$$

(iv) Write

$$(a + a^\dagger)^2\psi_n$$

in terms of ψ_{n-2}, ψ_n and ψ_{n+2}. Hence show that

$$\langle\psi_n|\hat{p}^4|\psi_n\rangle = \frac{3}{4}\hbar^2m^2\omega^2(2n^2 + 2n + 1)\,.$$

[*You may find the following identity useful:*

$$[A, BC] = B[A, C] + [A, B]C$$

for operators A, B and C.

[20 marks]

5. The angular momentum operators L_i ($i = 1, 2, 3$) satisfy the commutation relations $[L_1, L_2] = i\hbar L_3$ (and cyclic permutations), which imply that
$$[\mathbf{L}^2, L_1] = [\mathbf{L}^2, L_2] = [\mathbf{L}^2, L_3] = 0$$
(where $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$).
Suppose that $|l, m\rangle$ are the normalised eigenstates such that
$$L_3|l, m\rangle = \hbar m|l, m\rangle, \qquad \mathbf{L}^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle,$$
where $2l$ is a positive integer and the possible values of m are $-l, -l+1, \ldots l-1, l$. Here, $\mathbf{L}^2$ and L_3 form a complete commuting set of observables, so that these properties define $|l, m\rangle$ uniquely.

   (i) Defining $L_+ = L_1 + iL_2$ and $L_- = L_1 - iL_2$, show that
   $$[L_3, L_+] = \hbar L_+ \,, \qquad [L_3, L_-] = -\hbar L_- \,.$$
   Hence, also using the commutation relations for $\mathbf{L}^2$ above, deduce that
   $$L_+|l, m\rangle = M_{l,m}|l, m+1\rangle \quad \text{and} \quad L_-|l, m\rangle = N_{l,m}|l, m-1\rangle,$$
   where $M_{l,m}$ and $N_{l,m}$ are real, positive constants.

   (ii) A particle is in the normalised angular momentum state
   $$|\psi\rangle = z|1, -1\rangle + z^*|1, 1\rangle + c|1, 0\rangle$$
   where $z = a + ib$ and a, b and c are real.
   Show that the expectation value $\langle L_3\rangle$ of L_3 in this state is zero.
   By expressing L_1 in terms of L_+ and L_-, find the expectation value $\langle L_1\rangle$ for this state in terms of a, b and c.
   [*You may assume that in (i), $M_{l,m}$ and $N_{l,m}$ are given by*
   $$M_{l,m} = \hbar\sqrt{l(l+1) - m(m+1)} \quad \textit{and} \quad N_{l,m} = \hbar\sqrt{l(l+1) - m(m-1)}\,.]$$

[20 marks]

6. A particle of mass m moves in three dimensions under the influence of a Coulomb potential

$$V(r) = -\frac{A}{r} \qquad \text{where} \qquad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

and A is a positive constant.

(i) The normalised wave function for the first excited state with zero angular momentum is

$$\psi(\mathbf{r}) = Be^{-\frac{1}{2}\beta r}(1 - \frac{1}{2}\beta r) \qquad \text{where} \qquad B = \sqrt{\frac{\beta^3}{8\pi}}\,.$$

Determine β in terms of m, A and $\hbar$, and show that the energy E is given by

$$E = -\frac{mA^2}{8\hbar^2}\,.$$

(ii) The particle is now subjected to an additional potential λr, where λ is a small parameter. Show that the energy of this state, to first order in λ, is given by $E + \delta E$ where

$$\delta E = 6\frac{\lambda\hbar^2}{mA}\,.$$

[*Standard results from perturbation theory may be assumed without proof. Moreover, you may assume that the radial part of the Laplacian in spherical polar co-ordinates is*

$$\nabla^2_{\text{rad}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\,,$$

and also that

$$\int_0^\infty r^n e^{-\beta r}\,dr = \frac{n!}{\beta^{n+1}} \qquad (\beta > 0)\,.]$$

[20 marks]

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7. A particle of mass m moves on the x-axis subject to a potential

$$V(x) = \eta |x|^3 ,$$

where η is a positive constant.

Consider a normalised wave function of the form

$$\psi(x) = \begin{cases} B(a^2 - x^2) & : \quad |x| \leq a \\ 0 & : \quad \text{otherwise}, \end{cases}$$

where B and a are real positive constants.

(i) Compute the normalisation constant B.

(ii) Show that with this wave function, the expectation value of the Hamiltonian is given by

$$\langle \hat{H} \rangle = \frac{5\hbar^2}{4ma^2} + \frac{5\eta a^3}{64} .$$

(iii) Hence use the variational principle to show that the ground state energy is at most

$$E_0^{\max} = \frac{25}{16}\Big(\frac{\hbar^6 \eta^2}{27m^3}\Big)^{\frac{1}{5}} .$$

How might you try to improve on this estimate of the ground state energy and how would you know if you had succeeded in finding a more accurate estimate?

[20 marks]