PAPER CODE NO.
MATH 322

THE UNIVERSITY
of LIVERPOOL

MAY 2006 EXAMINATIONS

Bachelor of Arts : Year 3
Bachelor of Arts : Year 4
Bachelor of Science : Year 3
Bachelor of Science : Year 4
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Physics : Year 4
Master of Science : Year 1
No qualification aimed for : Year 1

CHAOS THEORY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be taken into account.

1. (a) $f(x)$ is a continuous, differentiable and invertible function with domain $[0,1]$ and range $[0,1]$. $f(x)$ has only three fixed points: unstable fixed points at $x=0$ and 1, and a stable fixed point at $x=0.5$.

(i) Sketch the graph of $f(x)$. [2 marks]

(ii) Find the basin of attraction for the stable fixed point and indicate it on the graph. [1 marks]

(b) A tent map $f(x)$ for x in $[0,1]$ is defined as

$$f(x) = 3\mu x \quad \text{for} \quad x \leq \frac{1}{2}$$

$$f(x) = 3\mu(1-x) \quad \text{for} \quad x > \frac{1}{2}$$

where $0 \leq \mu \leq 1$.

Consider the dynamical system given by iterations of this map

$$x_{n+1} = f(x_n).$$

(i) Sketch the graph of the function $f(x)$ for the two cases $\mu < 1/3$ and $\mu > 1/3$. [3 marks]

(ii) Show that, provided that $\mu \neq 1/3$, the fixed points are $x^* = 0$ for any μ and $x^* = 3\mu/(1+3\mu)$ if $\mu > 1/3$. [7 marks]

(iii) Show that $x^* = 0$ is a stable fixed point for $\mu < 1/3$ and that $x^* = 3\mu/(1+3\mu)$ is an unstable fixed point. [7 marks]

2. (a) Consider the dynamical system obtained by iterating the map

$$f(x) = 1 - 2\mu x^2$$

for $x \in [-1,1]$ and $0 < \mu < 2$.

Show that one of the fixed points of the system is at $x^* = (-1+\sqrt{1+8\mu})/4\mu$ and show that this fixed point is stable if $\mu < 3/8$. [5 marks]

(b) Now investigate the properties of $f^{(2)}(x) = f(f(x))$.

(i) Show that this map has an additional fixed point at

$$x^* = \frac{1+\sqrt{8\mu-3}}{4\mu}.$$

[8 marks]

(ii) Show that x^* corresponds to a 2-cycle of $f(x)$ and that this is stable for $3/8 < \mu < 5/8$. [7 marks]

3. Consider the dynamical systems defined by iterations of a function $f(x)$ in the following *four* cases:

$$
\begin{array}{ll}
\text{(i)} & f(x) = 3x \pmod 1 \\
\text{(ii)} & f(x) = x + 2.1 \pmod 1 \\
\text{(iii)} & f(x) = \sqrt{3}x^2, \qquad 0 \le x \le 1 \\
\text{(iv)} & f(x) = 5x - 3x^3, \qquad x \in \mathbb{R}.
\end{array}
$$

(a) In each case, find any fixed points and determine their stability. [8 marks]

(b) For cases (i) and (ii), find the Lyapunov exponent and say what you can deduce from its value. [8 marks]

(c) For cases (i) and (ii), discuss the limiting behaviour as $n \to \infty$ and how this is affected by the starting value, x_0. [4 marks]

4. Consider the dynamical system $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$ generated by the functions

$$
\begin{aligned}
f(x, y) &= x^2 - y^2 + a \\
g(x, y) &= 3xy,
\end{aligned}
$$

where a is a constant.

(i) Show that the system has fixed points given by $x^* = \frac{1}{2}\left(1 \pm \sqrt{1-4a}\right)$, $y^* = 0$ for $a < 1/4$ and $x^* = 1/3$, $y^* = \pm\frac{1}{3}\left(\sqrt{9a-2}\right)$ for $a > 2/9$. [7 marks]

(ii) Linearize the system about the appropriate fixed points for $a < 2/9$ and show that the system has a stable fixed point for $-4/9 < a < 2/9$. [8 marks]

(iii) Consider the set of points on a circle of radius r centred at the origin. Show that they are mapped under one step of this dynamical system to an ellipse and sketch the ellipse for $a = 2, r = 1$. [5 marks]

5. Consider the dynamical system described by the equations

$$\frac{dx}{dt} = (1-x)(1-bx) + 2x^2y$$
$$\frac{dy}{dt} = bx(1-x) - 2x^2y,$$

where b is a real positive parameter.

(i) Find the fixed point of the system and discuss its stability for $b > 0$, with $b \neq 3$ and $b \neq 3 + 2\sqrt{2}$. [8 marks]

(ii) For the particular case of the system when $b = 4$, consider trajectories which pass through the four points $(1/4, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$ and sketch the directions of the tangents to these trajectories. [7 marks]

(iii) Give *plausibility only* arguments that the system has a stable limit cycle when $b \approx 4$. [3 marks]

(iv) Discuss whether this system exhibits chaotic behaviour. [2 marks]

6. (a) The Sierpinski carpet is constructed from a unit square by dividing the square into 3×3 smaller equal squares, removing the central smaller square to leave the 8 smaller squares around the perimeter, then repeating the procedure for these 8 squares, and so on.

(i) Sketch the first three levels of this process, starting with and including the unit square itself. [2 marks]

(ii) Find the capacity dimension of the resulting infinite set. [4 marks]

(b) A dynamical system on $[0, 1]$ is given by

$$x_{n+1} = f(x_n)$$

where

$$f(x) = 0 \quad \text{for} \quad \frac{1}{4} < x < \frac{3}{4}$$
$$f(x) = 4x \quad (\text{mod } 1), \quad \text{otherwise.}$$

(i) Sketch the graph of $f(x)$. [2 marks]

(ii) Show that the fixed points of this system are unstable. [2 marks]

(iii) Consider the set S of initial points x_0 for which $x_n \neq 0$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S. [8 marks]

(iv) Give an example in base 4 of an initial value x_0 for which the system will show periodic behaviour. [2 marks]

7. Consider the Lorenz system

$$\begin{aligned}\frac{dx}{dt} &= y - x\\ \frac{dy}{dt} &= \rho x - y - xz\\ \frac{dz}{dt} &= -z + xy\,,\end{aligned}$$

with ρ a real positive constant.

(i) Show that the origin is a fixed point, $P_1 = (0,0,0)$, and that its stability depends on eigenvalues λ satisfying

$$(\lambda+1)\left[\lambda^2 + 2\lambda + 1 - \rho\right] = 0\,.$$

[5 marks]

(ii) Deduce that this fixed point is stable only when $0 < \rho < 1$. [5 marks]

(iii) Show that there are two further fixed points

$$P_2\,,\ P_3 = \left\{\left(\pm(\rho-1)^{1/2}\,,\ \pm(\rho-1)^{1/2}\,,\ (\rho-1)\right)\right\}\,,$$

when $\rho > 1$ and that their stability depends on eigenvalues λ satisfying

$$\lambda^3 + 3\lambda^2 + (1+\rho)\lambda + 2(\rho-1) = 0\,.$$

[7 marks]

(iv) Show that, if $\bar{z} = z - \rho - 1$, then

$$\frac{1}{2}\frac{d}{dt}\left(x^2 + y^2 + \bar{z}^2\right) = -x^2 - y^2 - \left[\bar{z} + \frac{1}{2}(\rho+1)\right]^2 + \frac{1}{4}(\rho+1)^2\,,$$

so that $\sqrt{x^2 + y^2 + \bar{z}^2}$ decreases for all states outside a particular sphere (implying the existence of an attractor). [3 marks]