PAPER CODE NO.
MATH322

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MAY 2005 EXAMINATIONS

Bachelor of Arts: Year 3
Bachelor of Arts: Year 4
Bachelor of Science: Year 3
Bachelor of Science: Year 4
Master of Mathematics: Year 3
Master of Mathematics: Year 4
Master of Physics: Year 3
Master of Physics: Year 4
Master of Science: Year 1
No qualification aimed for: Year 1

CHAOS THEORY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1

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1. (a) Consider the logistic map given by iterations of a function

$$f(x) = Ax(1-x) \ , \ 0 \leq x \leq 1 \ , \qquad (1)$$

where A is strictly positive and real.
(i) Show that there is an attracting fixed point for $0 \leq A < 3$ (you need not discuss the case $A = 1$). [5 marks]
(ii) Show that there is a 2-cycle attracting fixed point for $3 < A < 1 + \sqrt{6}$. [5 marks]

(b) Show that the map given by iterations of the function

$$g(x) = 1 - ax^2 \ , \qquad (2)$$

is essentially the same as the logistic map, with $a = \frac{1}{4}A(A-2)$. [Hint: Starting with $y_{n+1} = g(y_n)$, you may consider the successive changes of variables $z_n = ay_n$, $u_n = z_n - Y$ and $x_n = -u_n/(2Y)$, where Y has to be chosen appropriately]. [7 marks]

(c) Deduce the link between the logistic map associated with (1) and the Mandelbrot set by expressing it in terms of 'complex a plane' for the equivalent map associated with (2). [3 marks]

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2. (a) Consider the dynamical system given by $x_{n+1} = f(x_n)$. Define a fixed point of this system and derive the general condition for its stability (you need not discuss the special cases that may arise when the general condition is inconclusive). [4 marks]

(b) Consider the dynamical systems given by iterations of the following three functions

$$
\begin{aligned}
f_1(x) &= x + 1/\sqrt{3} && \mathrm{mod}(1)\ , \\
f_2(x) &= \sqrt{3}x && \mathrm{mod}(1)\ , \\
f_3(x) &= x + 1/2 - K/2\pi \sin(2\pi x) && \mathrm{mod}(1)\ ,
\end{aligned}
$$

where K is a positive real constant.

(i) Let us first assume that $K = \pi$. Find any fixed points of all three functions and determine their stability (you need not discuss any further any inconclusive case). [6 marks]

(ii) Discuss the sensitivity to the initial condition for the first two functions by evaluating the Lyapunov exponent. [4 marks]

(iii) Discuss briefly how the irrational nature of $1/\sqrt{3}$ leads to a quasi-periodic behaviour for the dynamical system $x_{n+1} = f_1(x_n)$. [2 marks]

(iv) Assume finally that K is just below π ($K = \pi - \varepsilon$, ε small). Approximate $f_3(x)$ as a quadratic series about $x = 1/4$. Sketch the map $f_3(x)$ for x near $1/4$ and show that the dynamical system $x_{n+1} = f_3(x)$ may have intermittent behaviour since the number of steps spent near $1/4$ increases as $1/\sqrt{\varepsilon}$ as $\varepsilon \to 0$. [4 marks]

3. (a) Cantor's 'middle fifth' is produced by the iterated process of removing the open middle fifth segment from previous segments, starting with the unit interval.
(i) Find the sum $L(n)$ of the lengths of remaining intervals at the n^{th} stage of the iteration and show that $L(n) \to 0$ as $n \to \infty$. [3 marks]
(ii) Find the capacity dimension of this Cantor set. [7 marks]

(b) A dynamical system on $[0, 1]$ is given by

$$x_{n+1} = f(x_n) \; ,$$

where

$$f(x) = \begin{cases} 5x & , \text{for } 0 \leq x \leq \frac{1}{5} \, , \\ 5x - 2 & , \text{for } \frac{2}{5} < x < \frac{3}{5} \, , \\ 5x - 4 & , \text{for } \frac{4}{5} \leq x \leq 1 \, , \\ 0 & , \text{otherwise} \, . \end{cases}$$

(i) Sketch $f(x)$. Show that the fixed points of this system are unstable. [5 marks]
(ii) Consider the set S of initial points x_0 for which $x_n \neq 0$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S. [5 marks]

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4. Consider a dynamical system defined by iterates of the functional relationships

$$\begin{cases} x_{n+1} &= f(x_n, y_n) \,, \\ y_{n+1} &= g(x_n, y_n) \,, \end{cases}$$

with

$$\begin{cases} f(x,y) &= 1 + y - ax^2 \,, \\ g(x,y) &= x \,, \end{cases}$$

where $a > 0$.

(a) Show that the system is invertible by finding its inverse map. Consider the image of the x-axis under one iteration of the system, obtain an algebraic expression for this image and sketch it when $a = 1$. [5 marks]

(b) Now consider the system for all (x, y) and find all fixed points. Linearise the system about them and find their stability. Is the dynamical system dissipative, area-preserving or area-expending? [15 marks]

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5. (a) Discuss briefly some possible bifurcations that can occur as a parameter is varied in a dynamical system described by two autonomous coupled differential equations. [4 marks]

(b) Consider the equations describing the dynamical behaviour of a system

$$\begin{cases} \dfrac{dx}{dt} &= by(1-y) - y^2x \ , \\ \\ \dfrac{dy}{dx} &= (1-y)(1-by) + y^2x \ , \end{cases}$$

where b is a real positive parameter.

(i) Find the fixed point of the system and discuss briefly its stability for $b > 0$, $b \neq 2$ and $b \neq 4$. [10 marks]

(ii) For the particular case when $b = 3$, consider trajectories which pass through the four points $(0, 1/3)$, $(1, 1)$, $(0, 2)$ and $(-1, 1)$. Sketch the directions of the tangents to these trajectories. Give plausible arguments that the system has a stable limit cycle when $b = 3$. [6 marks]

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6. Consider the Lotka-Volterra system:

$$\begin{cases} \dfrac{dx}{dt} &= ax - bxy \ , \\[2ex] \dfrac{dy}{dt} &= -cy + dxy \ , \end{cases}$$

where a, b, c and d are positive constants and x, y stand for prey, predators respectively (*e.g.* rabbits, foxes).

(a) Discuss briefly the critical points $(0, 0$ and $(c/d, a/b)$ of this system and their nature. [7 marks]

(b) Draw the phase portrait of this system and deduce the main ecological consequences of the model. [5 marks]

(c) Using the substitution $x = e^p$, $y = e^q$, show that the above system takes on the Hamiltonian canonical form

$$\begin{cases} \dfrac{dp}{dt} &= -\dfrac{\partial H}{\partial q} \ , \\[2ex] \dfrac{dq}{dt} &= \dfrac{\partial H}{\partial p} \ , \end{cases}$$

where H is a function depending upon p and q. [4 marks]

(d) Discuss briefly whether or not such a 2-coupled autonomous first-order differential equations may exhibit chaos. [4 marks]

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7. Consider the Lorenz system

$$\begin{cases} \dfrac{dx}{dt} &= \sigma(y - x) \ , \\ \dfrac{dy}{dt} &= \rho x - y - xz \ , \\ \dfrac{dz}{dt} &= -\beta z + xy \ , \end{cases}$$

with σ, ρ, β some real positive constants.

(a) Show that the origin is a fixed point $P_1 = (0,0,0)$ and that its stability depends on eigenvalues λ satisfying

$$(\lambda + \beta)\left[\lambda^2 + (\sigma+1)\lambda + \sigma(1-\rho)\right] = 0 \ .$$

Deduce that P_1 is stable only when $0 < \rho < 1$. [10 marks]

(b) Show that there are two further critical points

$$P_2 \ , \ P_3 = \left\{ \pm[\beta(\rho-1)]^{1/2} \ , \ \pm[\beta(\rho-1)]^{1/2} \ , \ (\rho - 1) \right\} \ ,$$

when $\rho > 1$, and that their stability depends on eigenvalues λ satisfying

$$\lambda^3 + \lambda^2(\sigma + \beta + 1) + \lambda\beta(\sigma + \rho) + 2\sigma\beta(\rho - 1) = 0 \ .$$

[5 marks]

(c) Show that if $\bar{z} = z - \rho - \sigma$, then

$$\frac{1}{2}\frac{d}{dt}\left(x^2 + y^2 + \bar{z}^2\right) = -\sigma x^2 - y^2 - \beta\left[\bar{z} + \frac{1}{2}(\rho+\sigma)\right]^2 + \frac{1}{4}\beta(\rho+\sigma)^2 \ ,$$

so that $\sqrt{x^2 + y^2 + \bar{z}^2}$ decreases for all states outside a particular ellipsoid (implying the existence of an attractor). [5 marks]