SUMMER 2004 EXAMINATIONS

Bachelor of Arts	:	Year 3
Bachelor of Arts	:	Year 4
Bachelor of Science	:	Year 3
Bachelor of Science	:	Year 4
Master of Mathematics	:	Year 3
Master of Mathematics	:	Year 4
Master of Physics	:	Year 3
Master of Physics	:	Year 4

CHAOS THEORY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Consider the dynamical system given by $x_{n+1} = f(x_n)$. Give a brief discussion of the period-doubling route to chaos, mentioning the role of the Feigenbaum constants. [7 marks]

For the system defined by $f(x) = -x^3 + \mu x$, find the fixed points of the system and determine their stability for all values of the parameter μ. [10 marks]

For the particular value $\mu = 3/2$, calculate the Lyapounov exponent and discuss the sensitivity of this system to the initial condition. [3 marks]

2. Consider the dynamical system given by iterations of a function

$$f(x) = 1 - \mu x^2, \qquad 0 \le x \le 1, \quad 0 < \mu \le 1.$$

Show that there is a fixed point at

$$x^* = \frac{-1 + \sqrt{1 + 4\mu}}{2\mu}.$$

Find the values of μ for which this fixed point is stable. [7 marks]

Show that there is a 2-cycle when $-\mu^2 x^2 + \mu x + \mu - 1 = 0$. For what values of μ does this have real solutions for x? [10 marks]

Give an argument why the system defined by $f(x)$ will not exhibit chaos.

[3 marks]

3. Give a brief discussion of quasi-periodic behaviour. [6 marks]

Consider the dynamical system defined by iterations of the following function

$$f(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x) \qquad \mathrm{mod}(1),$$

where K and Ω are real and $K \ge 0$. Define the winding number of a solution of this system. Show that the system has a stable fixed point for

$$2\pi|\Omega| < K < \sqrt{4 + (2\pi\Omega)^2},$$

and give the winding number of this solution. [12 marks]

Discuss the behaviour of the system when $K = 0$ for the two cases $\Omega = 1/2$ and $\Omega = \sqrt{3}/2$. [2 marks]

4. Define the capacity dimension of a set. Define the concept of self-similarity, and give an example of a set that is self-similar under rescaling. [6 marks]

A dynamical system defined on $x \in [0, 1]$ is given by $x_{n+1} = f(x_n)$ where

$$f(x) = 0, \qquad 1/10 \leq x \leq 9/10$$

$$f(x) = 10x \quad \text{mod}(1) \text{ otherwise.}$$

Sketch $f(x)$. Show that the fixed points of this system are unstable. Consider the set S of points for which x_n does not tend to zero. Obtain a description of S and calculate its capacity dimension. [10 marks]

Give a description of the initial values x_0 for which the system exhibits i)chaos ii)a 4-cycle. [4 marks]

5. Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n) \ , \qquad y_{n+1} = g(x_n, y_n)$$

with

$$f(x, y) = \frac{1}{2}x^2 + ay^2 \ , \qquad g(x, y) = xy \ .$$

Consider the system for all real values of the parameter a and find all fixed points. Linearise the system about them and discuss their stability. [15 marks]

On a single diagram, sketch the trajectories near each of the fixed points. [You do not need to calculate the eigenvectors.] [5 marks]

6. (a) Consider the dynamical system described by

$$\frac{dx}{dt} = x - y - xy^2$$

$$\frac{dy}{dt} = x.$$

Find any fixed points and determine their stability. [7 marks]

Consider the trajectories as they pass through the four points $(x, y) = (\pm 1, \pm 1)$ and sketch the tangents to the trajectories. Discuss how a Poincare section could be used to investigate the nature of the solution. [6 marks]

(b) Discuss briefly the concept of critical behaviour with reference to the sandpile model. [7 marks]

7. Consider the equations

$$\frac{dx}{dt} - y - x$$

$$\frac{dy}{dt} = xz - y$$

$$\frac{dz}{dt} = a - xy - z$$

where a is a real, positive parameter.

Show that this system of equations describes a dissipative system. [2 marks]

Find the fixed points of these equations. [6 marks]

Determine the stability of the fixed points for all values of a. [You can use the fact that the roots of a cubic $x^3 + \alpha x^2 + \beta x + \gamma$ all have real parts less than 0 if $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\alpha\beta > \gamma$.] [10 marks]

For the particular value $a = 3$, describe the nature of the fixed point at $(0, 0, 3)$. [2 marks]