PAPER CODE NO.
MATH322

SUMMER 2003 EXAMINATIONS

Bachelor of Arts : Year 3
Bachelor of Arts : Year 4
Bachelor of Science : Year 3
Bachelor of Science : Year 4
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Physics : Year 4

CHAOS THEORY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

---

1. Consider the dynamical system given by iterations of a function $f(x)$ defined on $x \in [0, 1]$, where

$$f(x) = 2x, \qquad 0 \leq x \leq \frac{1}{2}$$

$$f(x) = 2(1-x), \qquad \frac{1}{2} \leq x \leq 1.$$

Find the fixed points of this system and show that they are unstable. [5 marks]

Find $F(x) = f(f(x))$. Sketch $F(x)$ and find all of its fixed points. [12 marks]

Discuss the sensitivity of this system to the initial condition. [3 marks]

2. Consider the dynamical system given by iterations of a function $f(x)$. Define a fixed point of this system, and derive a general condition for it to be stable. (You need not discuss the case when the general condition is inconclusive.) What additional condition must be satisfied for the fixed point to be superstable? [6 marks]

(a) Consider the dynamical system given by iterations of a function

$$f(x) = 1 - \mu x^2, \qquad x \in [-1, 1]$$

where μ is a real, positive parameter. $F(x) = f(f(x))$ has a fixed point at

$$x^* = \frac{1 \pm \sqrt{4\mu - 3}}{2\mu}.$$

Show that this is a 2-cycle of f. [3 marks]

Find the value of μ for which this fixed point is superstable. [6 marks]

(b) Consider the fractal set generated by removing the middle 1/4 of the line segment $(0, 1)$, and then repeatedly removing the middle 1/4 of each remaining line segment. Find the capacity dimension of the resulting set. [5 marks]

3. Consider the dynamical system defined by iterations of a function $f(x)$ in the following *four* cases:

(i) $f(x) = 2x \pmod 1$

(ii) $f(x) = x + 2/\sqrt{3} \pmod 1$

(iii) $f(x) = x^2$, $0 \le x \le 1$

(iv) $f(x) = \frac{4}{3}x - 2x^3$, $x \in R$

In each case find any fixed points and determine their stability. [10 marks]

For cases (i) and (ii) find the Lyapounov exponent. For all four cases discuss the behaviour of the system as the number of iterations becomes large (this may depend on the starting value x_0). [10 marks]

4. Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n)\ , \qquad y_{n+1} = g(x_n, y_n)$$

with

$$f(x, y) = 1 + y - ax^2\ , \qquad g(x, y) = x\ , \qquad a > 0.$$

Show that this system is invertible by finding the inverse map. [3 marks]

Consider the system for all (x, y) and find all fixed points. Linearise the system about them and discuss their stability. [12 marks]

Define what it means for a two-dimensional dynamical system to be

(i) dissipative

(ii) area-preserving

(iii) area-expanding.

Classify the dynamical system above. [5 marks]

5. (a) Consider the dynamical system described by

$$\frac{dx}{dt} = (1-x)^2 + x^2 y$$

$$\frac{dy}{dt} = x(1-x) - x^2 y.$$

Show that the trajectories cannot cross in phase space. [2 marks] Find any fixed points and determine their stability. [8 marks]

(b) Show that the equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

can be written as two coupled first-order differential equations. Find the fixed point. Sketch the tangents to the trajectories passing through the four points $(x, y) = (\pm 1, \pm 1)$. Hence sketch the trajectories in phase space for the whole plane. Deduce the basin of attraction of the fixed point. [10 marks]

6. Consider the equations

$$\frac{dx}{dt} = 2y - 2x$$

$$\frac{dy}{dt} = ax - xz - y$$

$$\frac{dz}{dt} = xy - 2z$$

where a is a real, positive parameter.

Show that this system of equations describes a dissipative system [2 marks]

Show that the origin is a fixed point and find any other fixed points of these equations. [5 marks]

Consider the fixed point at the origin and determine its stability for all values of the parameter a. [6 marks]

For the particular value $a = 4/3$, determine the stability of the non-zero fixed points given that one of the eigenvalues is -1. [7 marks]

7. Discuss briefly all of the three following topics:

(i) A model for diffusion limited aggregation that exhibits fingering. [7 marks]

(ii) Intermittency. [6 marks]

(iii) The period-doubling route to chaos. [7 marks]