PAPER CODE NO.
MATH322

THE UNIVERSITY
of LIVERPOOL

SUMMER 2001 EXAMINATIONS

Degree of Bachelor of Science : Year 3
Degree of Master of Mathematics : Year 3
Degree of Master of Physics : Year 3

CHAOS THEORY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for FIVE complete answers.
Only the best FIVE answers will be taken into account.

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1. Consider the dynamical system given by iterations of a function:

$$x_{n+1} = f(x_n).$$

where the map f is $f(x) = 3\mu x - 3x^3, \quad x \in \mathbb{R}$.

(i) Find the fixed points. Show that there is only one fixed point for $\mu \le 1/3$ and there are three for $\mu > 1/3$.

2 marks

(ii) Using your results from part (i), determine stability of these fixed points for all values of the parameter μ.

18 marks

2. (a) Consider the dynamical system given by iterations of a function:

$$x_{n+1} = Q(x_n).$$

where the quadratic map Q is given by

$$Q(x) = 2x^2 - 0.5.$$

(i) Find the fixed points and determine their stability.

5 marks

(ii) Find the 2-periodic points and determine their stability.

10 marks

(b) Consider the dynamical system given by iterations of a function $x_{n+1} = x_n^{1/7}$. Find the Lyapunov exponent of the map and characterise whether the behaviour of the system is chaotic or regular.

5 marks

3. Consider a nonlinear dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n)$$

with $f(x, y) = 1.4x - 0.2x^2 - 0.1xy, \quad g(x, y) = 1.4y - 0.2y^2 - 0.1xy$.

(i) Find all equilibrium vectors of the system.

(ii) Study the stability of the equilibrium vectors.

20 marks

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4. (a) In the plane $0xy$, consider a figure $ABCDEFGHI$ of 8 subintervals whose ends A, B, C, D, E, F, G, H, I have the following coordinates $(0,0)$, $(1/4,0)$, $(1/4,1/4)$, $(2/4,1/4)$, $(2/4,0)$, $(2/4,-1/4)$, $(3/4,-1/4)$, $(3/4,0)$, $(1,0)$, respectively.

Consider the operation Q of exchanging of an interval of size L by a copy of the figure $ABCDEFGHI$ with scaling factor $\lambda = L$, i.e. if the length of an interval is $1/4$ then it is replaced by the 4 times reduced figure $ABCDEFGHI$.

Apply Q initially to the interval AB and then to each of other 7 intervals, i.e. first to BC and so on until HI. Then apply Q again to each of these smaller subintervals. The process is repeated infinitely.

Discuss whether the resulting set is self-similar under the discrete group of coordinate dilation. If yes then find its scaling factor.

4 marks

Find box-counting dimension of the set.

5 marks

(b) A dynamical system defined on $[0,1]$ is given by

$$\begin{aligned} x_{n+1} &= f(x_n) && \text{where} \\ f(x) &= 4x && \text{for} \quad 0 \le x < \frac{1}{4} \\ f(x) &= 4x-2 && \text{for} \quad \frac{1}{2} < x < \frac{3}{4} \\ f(x) &= 0 && \text{otherwise.} \end{aligned}$$

Sketch $f(x)$.

1 mark

Show that the fixed points of this system are unstable.

2 mark

Consider the set S of initial points x_0 for which x_n is not eventually 0. Obtain a description of S and use it to find the box-counting dimension of S.

8 marks

5. Consider the Weierstrass-Mandelbrot function for $x \geq 0$

$$F(x;p) = \sum_{n=-\infty}^{\infty} p^{(D-2)n}(1 - \cos p^n x), \quad p > 1, \quad 1 < D < 2.$$

(i) Show that $F(x;p)$ is a parametric-homogeneous function of degree $d = (2 - D)$ and parameter p.

3 marks

(ii) Give a formulation of the decomposition theorem for parametric-homogeneous functions. Applying the theorem, construct from $F(x;p)$ other fractal parametric-homogeneous functions: b_0 and b_4 of degree $d = 0$ and $d = 4$ respectively, having the same parameter p. Describe the scaling properties of the function b_4 and its trend.

7 marks

(iii) It is known that the box-counting dimension of the graph $F(x;p)$ is equal to D, calculate the box-counting dimension of the b_4 graph.

5 marks

(iv) It is known that $F(x;p)$ is bounded in the neighbourhood of $x = 0$. Show that the constructed function b_4 is differentiable from the right at $x = 0$ and find the derivative $(x \geq 0)$.

5 marks

6. (i) Discuss briefly some possible bifurcations that can occur as a parameter is varied in a dynamical system described by two autonomous coupled differential equations.

2 marks

Consider the equations describing the dynamical behaviour of a system

$$\frac{dx}{dt} = by - y^2 x,$$

$$\frac{dy}{dt} = 1 - (b+1)y + y^2 x$$

where b is a real positive parameter.

Find the fixed point of the system and discuss its stability for $b > 0$, and $b \neq 2$ and $b \neq 4$.

11 marks

(ii) For the particular case when $b = 3$, consider trajectories which pass through the four points $(2, 1)$, $(3, 2)$, $(3, 1/2)$, and $(4, 1)$ and sketch the directions of the tangents to these trajectories.

5 marks

Give plausibility arguments that the system has a stable limit cycle when $b = 3$.

2 marks

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7. Consider the equations

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = 2z - z(z - x)$$

$$\frac{dz}{dt} = -x - z + 2xy.$$

Show that the trajectories cannot cross in phase space.

2 marks

Show that this system of equations describes a dissipative system.

2 marks

Show that the origin is a fixed point and find the other fixed point of these equations.

4 marks

Consider first the fixed point at the origin: linearise the equations about this fixed point and show that one eigenvalue is 1 and find the others. Discuss the stability of this fixed point.

4 marks

For other fixed point: linearise the equations about it and find the characteristic equations for the eigenvalues. Show that one of these eigenvalues is -2 and find the others.

4 marks

Without evaluating the eigenvectors, describe briefly the motion of trajectories which are near each of these two fixed points.

4 marks