2MA62

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Consider the dynamical system given by iterations of a function

$$x_{n+1} = f(x_n).$$

Derive the general condition on $f(x)$ for a fixed point of a system to be stable.

Consider the function defined on the domain [0,1] as

$$f(x) = \mu x(1 - x^2)$$

where μ is a real non-negative constant and find the fixed points of the given system and the values of μ for which they are stable.

Consider now the system defined by $F(x) = f(f(x))$. Find for this system the values of μ for which there is a fixed point at $x^* = \frac{1}{\sqrt{3}}$.

Discuss the stability of this fixed point for each acceptable value of μ.

2. Consider a dynamical system defined by iteration of the function defined on the domain [0,1] as

$$f(x) = x + \Omega - \frac{K}{2\pi} \sin(2\pi x) \quad \mathrm{mod}(1),$$

where Ω and K are real constants.

Discuss the behaviour of this system for the two cases ($K = 0$, $\Omega = 0.5$) and ($K = 0$, $\Omega = 1/\sqrt{2}$).

From now on, consider the case where $\Omega = 0.5$ and K is close to π.

Show that the system for $K = \pi$ has a fixed point at $x = 0.25$ and find any other fixed points.

When K is just below π ($K = \pi - \epsilon$ with $0 < \epsilon << 1$), show that for x near 0.25, $f(x)$ can be approximated as a $x + \delta + a(x - 0.25)^2$ where you should find expressions for δ and a in terms of ϵ. Sketch the map $f(x)$ for x near 0.25 and show that the dynamical system may have intermittent behaviour since the number of steps spent near $x = 0.25$ increases as $1/\sqrt{\epsilon}$ as $\epsilon \to 0$.

3. (i) Consider a square divided into 4 smaller squares arranged as a 2×2 pattern. Consider the operation Q of removing the top left smaller square, leaving the other three. Apply Q initially to a unit length square to leave 3 smaller squares. Then apply Q again to each of these smaller squares. The process is repeated indefinitely.

Sketch the resulting set and discuss whether it is self-similar under scale changes and find its capacity dimension.

(ii) A dynamical system on [0,1] is given by

$$x_{n+1} = f(x_n) \quad \text{where}$$

$$f(x) = 4x \text{ for } x < 0.25$$

$$f(x) = 4x - 2 \text{ for } 0.5 < x < 0.75$$

$$f(x) = 1 \text{ otherwise.}$$

Sketch $f(x)$.

Consider any fixed points of this system with $x^* \neq 0,\ 1$ and show that they are unstable.

Consider the set S of initial points x_0 for which $x_n \not\to 1$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S.

4. Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n) \ , \qquad y_{n+1} = g(x_n, y_n)$$

with

$$f(x, y) = 1 + y - x^2 \ , \qquad g(x, y) = x \ .$$

Show that this system is invertible by explicitly obtaining the inverse map.

Consider the image under one iteration of the system of the straight line from $(x, y) = (-1, -1)$ to $(1, 1)$, obtain an algebraic expression for this image and sketch it.

Now consider the system for all (x, y) and find all fixed points. Linearise the system about them and discuss their stability.

Show that this is an area-preserving dynamical system.

5. Discuss briefly some possible bifurcations that can occur as a parameter is varied in a dynamical system described by two autonomous coupled differential equations.

Consider the equations describing the dynamical behaviour of a system

$$\frac{dx}{dt} = (1-x)(1-bx) + x^2 y$$
$$\frac{dy}{dt} = bx(1-x) - x^2 y,$$

where b is a real positive parameter.

Find the fixed point of the system and discuss its stability for all values of b.

For the particular case when $b = 3$, consider trajectories which pass through the four points $(1/3, 0)$, $(1, 1)$, $(2, 0)$ and $(1, -1)$ and sketch the directions of the tangents to these trajectories.

Give plausibility arguments that the system has a stable limit cycle when $b = 3$.

6. Consider the equations

$$\frac{dx}{dt} = 2y - y(y - z)$$
$$\frac{dy}{dt} = -z - y + 2zx$$
$$\frac{dz}{dt} = -x$$

Show that the trajectories cannot cross in phase space.

Show that this system of equations describes a dissipative system.

Show that the origin is a fixed point and find the other fixed point of these equations.

Consider first the fixed point at the origin: linearise the equations about this fixed point and show that one eigenvalue is 1 and find the others. Discuss the stability of this fixed point.

For the other fixed point: linearise the equations around it and find the characteristic equations for the eigenvalues. Show that one of these eigenvalues is -2 and find the others.

Without evaluating the eigenvectors, describe briefly the motion of trajectories which are near each of these two fixed points.

7. Discuss briefly all of the three following topics (the marks for each topic are the same):

(i) Express the equation for the forced, damped, circular pendulum in terms of autonomous coupled non-linear first-order differential equations and give arguments why chaotic solutions may exist.

(ii) A model that produces growth patterns showing fingering (eg. like snow-flakes) in 2 dimensions.

(iii) A model that is able to generate a power-law behaviour of the type that might be relevant to the study of earthquakes.