2MA62

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Consider the dynamical system given by iterations of a function

$$f(x) = 1 - \mu x^2$$

for $x \in [0, 1]$ and $0 < \mu < 1$.

Show that the system has a fixed point at $x^* = (-1 + \sqrt{1 + 4\mu})/(2\mu)$.

Find the region of stability of this fixed point.

For the case when $\mu = 3/4$, evaluate $f(x)$ and $F(x) = f(f(x))$ when $x = 0,\ 1/3,\ 2/3,\ 1$. Hence sketch the graphs of $f(x)$ and of $F(x)$.

Consider now the dynamical system defined by $F(x) = f(f(x))$. Show that the equation for a fixed point, which is not a fixed point of the system defined by $f(x)$, is given by $-\mu^2 x^2 + \mu x + \mu - 1 = 0$ and find the location of these fixed points when $\mu \geq 3/4$.

Give a brief argument why the system defined by $f(x)$ will not show chaos.

2. (i) With reference to a dynamical system $x_n = f(x_{n-1})$ where $f(x)$ is a function described by a parameter μ: describe the bifurcations that lead to (a) a period doubling route to chaos, and (b) intermittency.

(ii) Consider a dynamical system defined by iteration of the function defined on the domain [0,1] as

$$f(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x) \quad \mathrm{mod}(1),$$

where Ω and K are real constants and $K \geq 0$.

Define the winding number of a solution of this dynamical system.

Show that the system has a stable fixed point for

$$2\pi|\Omega| < K < \sqrt{4 + (2\pi\Omega)^2}$$

and give the winding number of this solution.

Discuss the behaviour of this system when $K = 0$ and for the two cases $\Omega = 1/3$ and $\Omega = \pi/10$.

3. (i) Consider a square divided into 9 smaller squares arranged as a 3×3 pattern. Consider the operation Q of removing 5 of these smaller squares, leaving the 4 at the centres of each side. Apply Q initially to a unit length square to leave 4 smaller squares. Then apply Q again to each of these smaller squares. The process is repeated indefinitely.

Discuss whether the resulting set is self-similar under scale changes and find its capacity dimension.

(ii) A dynamical system on [0,1] is given by

$$x_{n+1} = f(x_n) \quad \text{where}$$
$$f(x) = 0 \text{ for } \frac{2}{5} \le x \le \frac{3}{5}$$
$$f(x) = 5x \mod(1), \quad \text{otherwise.}$$

Sketch $f(x)$.

Show that the fixed points of this system are unstable.

Consider the set S of initial points x_0 for which $x_n \neq 0$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S.

Give an example in base 5 of an initial value x_0 for which the system will show periodic behaviour.

4. Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n)\ , \qquad y_{n+1} = g(x_n, y_n)$$

Derive the condition that a fixed point of the system is stable.

Consider the case

$$f(x, y) = \frac{x^2 + \mu}{2} - 2y^2$$
$$g(x, y) = xy\ ,$$

where μ is a real constant.

Find the fixed points of the system for $\mu < 1$.

Show that there is a stable fixed point when $-3 < \mu < 1$.

For the special case when $\mu = 0$, consider the segment of the straight line from $(x_0,\ y_0) = (1,\ 0)$ to $(1,\ 1)$. Find and sketch the set of points corresponding to this line segment after one iteration of the dynamical system.

5. Discuss briefly some possible bifurcations that can occur as a parameter is varied in a dynamical system described by two autonomous coupled differential equations.

Consider the dynamical system described by

$$\frac{d^2y}{dt^2} + (y^2 - \eta)\frac{dy}{dt} + y = 0 \ ,$$

where η is a real constant.

Express this equation as two coupled autonomous first-order differential equations.

Hence determine the stability of the fixed point at the origin.

For the particular case when $\eta = 1$, consider the trajectories as they pass through the four points $(x,\ y) = (\pm 1,\ 0),\ (0,\ \pm 1)$ and sketch the directions of the tangents to the trajectories.

Discuss how a Poincaré section might be used to investigate the nature of the solution for this case.

6. Consider the equations

$$\frac{dx}{dt} = -x + y$$

$$\frac{dy}{dt} = xz - y$$

$$\frac{dz}{dt} = a - z - xy \ ,$$

where a is a real positive constant.

Show that this system of equations describes a dissipative system.

Find the fixed points of these equations.

Establish the region of a, if any, for which each fixed point is stable.

[You may use the result that the roots of $x^3 + \alpha x^2 + \beta x + \gamma = 0$ when $\alpha > 0,\ \beta > 0,\ \gamma > 0$ and $\alpha\beta > \gamma$ satisfy $\mathrm{Re}\, x < 0$.]

Without evaluating the eigenvectors, describe briefly the motion of trajectories which are near any stable fixed points for $a = 2$.

7. Discuss briefly all of the three following topics:

(i) Why N coupled autonomous first-order differential equations can only show chaos for $N \geq 3$. Also discuss which other types of long-time behaviour are possible for $N = 3$.

(ii) The concept of critical behaviour, including a model that generates critical behaviour.

(iii) A model that produces growth patterns showing fingering (eg. like snow-flakes) in 2 dimensions.