**2MA62**

**Instructions to candidates**

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

**1.** Consider the dynamical system given by iterations of a function

$$x_{n+1} = f(x_n).$$

Derive the general condition on $f(x)$ for a fixed point of a system to be stable.

Consider the function defined on the domain [0,1] as

$$f(x) = \mu x(1 - x^2)$$

where $\mu$ is a real non-negative constant and find the fixed points of the given system. Show that a stable fixed point exists for $0 < \mu < 2$.

Find the value of $\mu$ for which the fixed point is super-stable ($df/dx = 0$).

Consider now the system defined by $f(f(x))$. Show that this system has a superstable fixed point when $\mu = \frac{3}{\sqrt{2}}$.

**2.** With reference to a dynamical system $x_n = f(x_{n-1})$ where $f(x)$ is a function described by a parameter $\mu$:

(i) Describe what is meant by chaos.

(ii) Assuming the system has a period doubling sequence for a range of values of $\mu$, sketch the expected behaviour versus $\mu$ of the fixed points $x^*_m$ of the $2^m$ cycles of the map. Define the Feigenbaum numbers $\alpha$ and $\delta$.

Obtain the approximate solution $\alpha = 1 + \sqrt{3}$ from Feigenbaum's equation $g(-\alpha z) = -\alpha g(g(z))$.

(iii) Show that the shift map

$$x_{n+1} = 2x_n \bmod 1$$

can exhibit both periodic and chaotic behaviour and specify the conditions under which these occur. Find the Lyapounov exponent of the map.

(iv) Sketch a function $f(x)$ such that the dynamical system will show intermittent behaviour. Describe why the behaviour will be intermittent.

**3.** (i) Consider an equilateral triangle divided into 4 smaller equilateral triangles by joining the midpoints of the sides. Consider the operation $Q$ of removing the central smaller triangle. Apply $Q$ initially to a unit length triangle to leave 3 smaller triangles. Then apply $Q$ again to each of these smaller triangles. The process is repeated indefinitely.

Discuss whether the resulting set is self-similar under scale changes and find its capacity dimension.

(ii) A dynamical system on [0,1] is given by

$$x_{n+1} = f(x_n) \quad \text{where}$$

$$f(x) = 3x \text{ for } 0 \leq x \leq \frac{1}{3}$$

$$f(x) = 3x - 2 \text{ for } \frac{2}{3} \leq x \leq 1$$

$$f(x) = 1 \text{ otherwise.}$$

Sketch $f(x)$.

Find the fixed points of this system and show that they are unstable.

Consider the set S of initial points $x_0$ for which $x_n \neq 1$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S.

Give an example in base 3 of an initial value $x_0$ for which the system will show periodic behaviour.

**4.** Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n)\ , \qquad y_{n+1} = g(x_n, y_n)$$

with

$$f(x, y) = x - \frac{K}{2\pi} \sin 2\pi y \quad \mathrm{mod}(1),$$

$$g(x, y) = y + x - \frac{K}{2\pi} \sin 2\pi y \quad \mathrm{mod}(1),$$

where $K$ is a real non-negative constant.

Find the Jacobian matrix of the system and show that the system is area preserving.

Consider the fixed point at the origin. Show that the eigenvalues of the Jacobian matrix satisfy $\lambda^2 - (2 - K)\lambda + 1 = 0$. Hence deduce that this fixed point is unstable for $K > 4$.

Find another fixed point of the system which exists for all $K$ and show that it is unstable for all $K > 0$.

For the special case when $K = 0$, discuss the behaviour of the dynamical system for different initial values $(x_0, y_0)$.

**5.** Discuss briefly the invariant sets that can occur in a dynamical system described by two autonomous coupled differential equations.

Consider the equations describing the dynamical behaviour of a system

$$\frac{dx}{dt} = y - ax(x^2 + y^2 - 1)$$

$$\frac{dy}{dt} = -x - ay(x^2 + y^2 - 1),$$

where $a$ is a real non-negative parameter.

Show that the fixed point at the origin is unstable for $a > 0$.

For the particular case when $a = 1$, consider the trajectories as they cross the line $x = 0$ and sketch the directions of the tangents to the trajectories which cross this line at $y = 0.5$, $1.0$ and $1.5$.

For the case when $a = 0$, these equations describe a harmonic oscillator and, defining $C(x, y) = x^2 + y^2$, evaluate $dC/dt$ using these equations to show that $C(x, y)$ is a constant of the motion.

For the general case when $a > 0$, use the equations to express $dC/dt$ in terms of $C$ and hence deduce that there is a stable solution with $C(x, y) = 1$. Sketch the trajectories in phase space on and near this stable solution.

**6.** Consider the equations

$$\frac{dx}{dt} = -z$$

$$\frac{dy}{dt} = -x - y + 2zx$$

$$\frac{dz}{dt} = 2y - y(y - x)$$

Show that this system of equations describes a dissipative system.

Show that the trajectories cannot cross in phase space.

Show that the origin is a fixed point and find the other fixed point of these equations.

Consider first the fixed point at the origin: linearise the equations about this fixed point and show that one eigenvlaue is 1 and find the others. Discuss the stability of this fixed point.

For the other fixed point: linearise the equations around it and find the characteristic equations for the eigenvalues. Show that these eigenvalues are $-2$ and $(1 \pm \sqrt{3}i)/2$.

Without evaluating the eigenvectors, describe briefly the motion of trajectories which are near each of these two fixed points.

**7.** Describe an algorithm for a discrete time model of diffusive aggregation in two dimensions. Show that the model has the property that fingering occurs by considering the probabilities of aggregation to a straight line of several sites. Indicate the expected patterns which will be formed for aggregation from an initial single site.

Discuss self organised criticality with particular reference to models of earthquakes.