MATH298 May 2006 exam: solutions (part A)

All problems are similar to homework and class examples, except where stated explicitly as bookwork.

1. (a) **Question** *Find the adjoint,* $\mathrm{adj}\,(\mathbf{A})$, *determinant,* $\det \mathbf{A}$, *and inverse,* $\mathbf{A}^{-1}$, *of the square matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix}.$$

Answer The minors are:

$$\mathrm{minors}\,(\mathbf{A}) = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} \\ \\ \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ -2 & -1 \end{vmatrix} \\ \\ \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ -1 & 2 & -4 \\ 3 & -1 & 2 \end{bmatrix}$$

The cofactors are:

$$\mathrm{cof}\,(\mathbf{A}) = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

The adjoint matrix is:

$$\mathrm{adj}\,(A) = \mathrm{cof}\,(\mathbf{A})^T = \begin{bmatrix} -1 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

Check by multiplying:

$$\mathbf{A}\mathrm{adj}\,(\mathbf{A}) = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = -5\mathbf{I},$$

thus

$$\det(\mathbf{A}) = -5$$

and

$$\mathbf{A}^{-1} = -\frac{1}{5}\mathrm{adj}\,(\mathbf{A}) = \begin{bmatrix} 0.2 & -0.2 & -0.6 \\ -0.6 & -0.4 & -0.2 \\ -0.2 & -0.8 & -0.4 \end{bmatrix}$$

[7]

(b) **Question** *Using your result from part (1a), find the solution to the system of simultaneous equations*

$$\begin{aligned} -2y + z &= -3, \\ x + y - 2z &= -1, \\ -2x - y + z &= 1. \end{aligned}$$

Answer

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b};$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.2 & -0.2 & -0.6 \\ -0.6 & -0.4 & -0.2 \\ -0.2 & -0.8 & -0.4 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

[6]

(c) **Question** *Solve the same system of equations as in part (1b), using Cramer's rule.*

Answer

$$\Delta = \begin{vmatrix} 0 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{vmatrix} = -5, \qquad \Delta_x = \begin{vmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{vmatrix} = 5,$$

$$\Delta_y = \begin{vmatrix} 0 & -3 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & 1 \end{vmatrix} = -10, \qquad \Delta_z = \begin{vmatrix} 0 & -2 & -3 \\ 1 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -5$$

Thus

$$x = \Delta_x/\Delta = -1, \quad y = \Delta_y/\Delta = 2, z = \Delta_z/\Delta = 1.$$

[4]

Total for this question: 17 marks

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2. (a) *Use Gaussian elimination method to show that one of the following systems has no solutions and the other has an infinite number of solutions. (HINT: find the rank of* **A** *and* **A**|**b** *and compare with* n*, the number of unknowns):*

(i) $\begin{bmatrix} 2 & -3 & -1 \\ 2 & 1 & -3 \\ 4 & -4 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$

Answer

$$\left[\begin{array}{ccc|c} 2 & -3 & -1 & 1 \\ 2 & 1 & -3 & 3 \\ 4 & -4 & -3 & 8 \end{array}\right] \xrightarrow{\begin{pmatrix} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{pmatrix}} \left[\begin{array}{ccc|c} 2 & -3 & -1 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 2 & -1 & 6 \end{array}\right] \xrightarrow{\left(r_3 \to r_3 - \frac{1}{2}r_2 \right)} \left[\begin{array}{ccc|c} 2 & -3 & -1 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

We have $\operatorname{rank}(\mathbf{A}) = 2 \neq \operatorname{rank}(\mathbf{A}|\mathbf{b}) = 3$. Therefore, the system $\mathbf{Ax} = \mathbf{b}$ has no solutions.

[7]

(ii) **Question** $\begin{bmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 4 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$

Answer

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 3 \\ 2 & -3 & 1 & 2 \\ 4 & 0 & 5 & 7 \end{array}\right] \xrightarrow{\begin{pmatrix} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{pmatrix}} \left[\begin{array}{ccc|c} 2 & -1 & 2 & 3 \\ 0 & -2 & -1 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\left(r_3 \to r_3 + r_2 \right)} \left[\begin{array}{ccc|c} 2 & -1 & 2 & 3 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b}) = 2$. Therefore, the system $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solutions.

[7]

(b) **Question** *Find the general solution of whichever of the above systems is consistent and write the solution in parametric form.*

Answer The general solution of the system (ii) in parametric form is: $z = c$, $y = \frac{1}{2} - \frac{1}{2}c$, $x = \frac{7}{4} - \frac{5}{4}c$.

[3]

Total for this question: 17 marks

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3. *Consider the matrix*

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

- **Question** *Write down its characteristic polynomial.*
  Answer Characteristic polynomial:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{bmatrix} -1-\lambda & 1 & 3 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}$$

$$= (-1-\lambda)\left((2-\lambda)^2 - 1\right) = -\lambda^3 + 3\lambda^2 + \lambda - 4 = 0.$$

  Question *One of its eigenvalues is $\lambda_1 = -1$. Find the other two, λ_2 and λ_3.*
  Answer By method of undetermined coefficients or by inspection (see above derivation) we can factorize

$$\lambda^3 - 3\lambda^2 - \lambda + 4 = (\lambda + 1)(\lambda^2 - 4\lambda + 3)$$

  hence the other two eigenvalues are roots of $\lambda^2 - 4\lambda + 3 = 0$, i.e. $\lambda = 1$ and $\lambda = 3$.

[5]

- **Question** *Find an eigenvector for each of the three eigenvalues λ_1, λ_2, λ_3.*
  Answer For $\underline{\lambda_1 = -1}$, equation $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  hence we may choose

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  For $\underline{\lambda_2 = 1}$, equation $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \Rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  hence we may choose

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\underline{\lambda_3 = 3}$, equation $(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} -4 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{ccc|c} -4 & 1 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} -4 & 1 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

hence we may choose

$$\mathbf{v}_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{[6]}$

- **Question** *Find the unit length (normalised) eigenvectors corresponding to the three eigenvalues λ_1, λ_2 and λ_3.*

  Answer The length of $\mathbf{v}_1$ is $\sqrt{1^2+0+0} = 1$, so $\mathbf{v}_1^{norm} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, the length of $\mathbf{v}_2$ is $\sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$, so $\mathbf{v}_2^{norm} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, the length of $\mathbf{v}_3$ is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$, so $\mathbf{v}_3^{norm} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\boxed{[3]}$

- **Question** *Consider the following system of ordinary differential equations:*

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -x + y + 3z \\ \mathrm{d}y/\mathrm{d}t &= 2y + z \\ \mathrm{d}z/\mathrm{d}t &= y + 2z \end{aligned}$$

  Using your previous results, write down a general solution of this system, depending on three arbitrary contants C_1, C_2 and C_3.

  Answer This system is equivalent to $\mathrm{d}\mathbf{u}/\mathrm{d}t = \mathbf{A}\mathbf{u}$ where $\mathbf{u} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and $\mathbf{A}$ is the matrix considered in the first part. Thus the solution is

$$\mathbf{u} = C_1\mathbf{v_1}e^{\lambda_1 t} + C_2\mathbf{v_2}e^{\lambda_2 t} + C_3\mathbf{v_3}e^{\lambda_3 t}$$

$$= C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + C_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{t} + C_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  or, by components

$$\begin{aligned} x &= C_1e^{-t} + C_2e^{t} + C_3e^{3t} \\ y &= -C_2e^{t} + C_3e^{3t} \\ z &= C_2e^{t} + C_3e^{3t} \end{aligned}$$

$\boxed{[3]}$

Total for this question: 17 marks

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4. (a) **Question** *Find the stationary point of the function*

$$f(x,y) = x^2 + 2y^2 + 3xy - 2x - 3y$$

Answer First derivatives test. The partial derivatives are

$$f_x = 2x + 3y - 2; \qquad f_y = 3x + 4y - 3.$$

The stationary point is the solution of equation $f_x = 0$, $f_y = 0$, which is $(x,y) = (1,0)$.
Question *Classify this point.*
Answer Second derivatives test. The second partial derivatives are

$$f_{xx} = 2; \qquad f_{xy} = 3; \qquad f_{yy} = 4y.$$

The determinant of their matrix is then

$$D = \begin{vmatrix} f_{xx} & f_{xx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1$$

Since $D < 0$, this is a saddle point.

[9]

(b) **Question** *Find an equation of the tangent plane at the point $(x,y) = (1,1)$ for the graph of the function $f(x)$ defined above in part (4a).*
Answer The value of the function and its partial derivatives at the given point are

$$f = x^2 + 2y^2 + 3xy - 2x - 3y = 1, \qquad f_x = 2x + 3y - 2 = 3, \qquad f_y = 3x + 4y - 3 = 4$$

thus an equation of the tangent plane is

$$z = 1 + 3(x-1) + 4(y-1) = -6 + 3x + 4y$$

Question *Find a normal vector to that plane.*
Answer

$$\mathbf{n} = \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

Question *Calculate also the directional derivative of this function in the direction $\theta = -45°$ at the same point $(1,1)$.*
Answer

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}s} &= \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y} \\ &= -\sqrt{2}/2 \end{aligned}$$

[8]

Total for this question: 17 marks

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5. (a) **Question** *Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} of the function*

$$f(x,y) = e^{2x} \sin y.$$

Answer

$$f_x = 2e^{2x} \sin y, \qquad f_y = e^{2x} \cos y$$
$$f_{xx} = 4e^{2x} \sin y, \quad f_{xy} = 2e^{2x} \cos y, \quad f_{yy} = -e^{2x} \sin y,$$

[4]

(b) **Question** *Using your result from part (5a), find the Taylor series at $(0, \pi/2)$ for f up to and including terms quadratic in the increments δx and δy.*

Answer Values of derivatives at the point $(0, \pi/2)$:

$$f_x = 2, \qquad f_y = 0$$

$$f_{xx} = 4, \quad f_{xy} = 0, \quad f_{yy} = -1,$$

Taylor's formula for the quadratic approximation

$$f(x+\delta x, y+\delta y) \approx f(x,y) + f_x(x,y)\delta x + f_y(x,y)\delta y + \frac{1}{2}\left(f_{xx}(x,y)(\delta x)^2 + 2f_{xy}\delta x\delta y + f_{yy}(x,y)(\delta y)^2\right)$$

Substituting here $x = 0$, $y = \pi/2$ and the value of the function and its derivatives at this point, obtain

$$f(\delta x, \pi/2 + \delta y) \approx 1 + 2\delta x + 2(\delta x)^2 - \frac{1}{2}(\delta y)^2$$

[9]

(c) **Question** *Use the approximation to the Taylor series found in part (5b) to obtain linear and quadratic approximation for $f(0.1, 1.5)$, with 3 decimal places.*

Answer These values are obtained for $\delta x = 0.1$ and $\delta y = 1.5 - \pi/2 \approx -0.0708$. Linear approximation:

$$f(0.1, 1.5) \approx 1 + 2\delta x = 1 + 2 * 0.1 = 1.2$$

Quadratic approximation:

$$f(0.1, 1.5) \approx 1 + 2\delta x + 2(\delta x)^2 - 0.5(\delta y)^2 \approx 1 + 2*0.1 + 2*0.01 - 0.5*(0.0708^2) \approx 1.217(4)$$

(cf the exact value $f(0.2, 1.5) = 1.2183\ldots$ — optional self-control, no credit)

[4]

Total for this question: 17 marks

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6. (a) **Question** *The function $g(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion*

$$g(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

Answer

$$a_0 = \frac{1}{2}\int_{-1}^{1} g(x)\,\mathrm{d}x,$$

$$a_n = \int_{-1}^{1} g(x)\cos(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2\ldots,$$

$$b_n = \int_{-1}^{1} g(x)\sin(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2\ldots$$

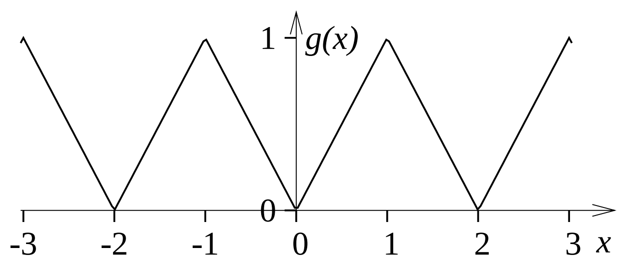
[5]

(b) **Question** *The function $g(x)$ is defined by*

$$g(x) = \begin{cases} -2x, & -1 \le x \le 0, \\ 2x, & 0 \le x \le 1, \\ g(x \pm 2), & \textit{for all } x. \end{cases}$$

Sketch the graph of $g(x)$ for $-3 < x < 3$.

Answer



Question *Give the definition of an even function.*

Answer $f(-x) = f(x)$ for all x.

Question *Explain what special features a Fourier series of an even function has.*

Answer It lacks all sin terms.

Question *Explain why the function $g(x)$ defined above is even.*

Answer Graphical: the graph is mirror-symmetric about the vertical axis.
Analytical: it is even by definition within the symmetric interval $-1 \le x \le 1$, and periodic with period 2 equal to the length of that interval, therefore even everywhere.

[4]

(c) **Question** *Find the Fourier series of the function $g(x)$ defined above. You may use the following result:* $\int x\cos(kx)\,\mathrm{d}x = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx)$, *where $k \neq 0$ is a constant.*

Answer

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} g(x)\,\mathrm{d}x$$

$$= \frac{1}{2} * 2\int_{0}^{1} 2x\,\mathrm{d}x$$

$$= 2\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \int_{-1}^{1} g(x)\cos(n\pi x)\,\mathrm{d}x$$

$$= 2\int_{0}^{1} g(x)\cos(n\pi x)\,\mathrm{d}x = 2\int_{0}^{1} 2x\cos(n\pi x)\,\mathrm{d}x$$

$$= 4\left(\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\right)_0^1 = 4\,\frac{1}{n^2\pi^2}\,(\cos(n\pi) - 1)$$

$$= -\frac{4}{n^2\pi^2}\,(1 - (-1)^n) = \begin{cases} \frac{8}{n^2\pi^2}, & \text{for odd } n, \\ 0, & \text{for even } n, \end{cases}$$

Thus,

$$g(x) = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5\ldots}\frac{1}{n^2}\cos(n\pi x)$$

Question *Write out this series explicitly up to terms with* $\cos(5\pi x)$ *and* $\sin(5\pi x)$*.*

Answer

$$g(x) \approx 1 - \frac{8}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x)\right)$$

[8]

Total for this question: 17 marks
