MATH298 May 2005 exam: solutions (part A)

All problems are similar to homework or class examples, except where stated explicitly as bookwork.

1. (a) *Use Gaussian elimination method to show that one of the following systems has no solutions and the other has an infinite number of solutions. (HINT: find the rank of* $\mathbf{A}$ *and* $\mathbf{A}|\mathbf{b}$ *and compare with* n*, the number of unknowns):*

(i) $$\begin{bmatrix} 1 & -4 & 1 \\ 1 & -2 & -1 \\ 2 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}.$$

Answer

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & -2 \\ 1 & -2 & -1 & 4 \\ 2 & -7 & 1 & -1 \end{array}\right] \xrightarrow{\left(\begin{array}{c} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{array}\right)} \left[\begin{array}{ccc|c} 1 & -4 & 1 & -2 \\ 0 & 2 & -2 & 6 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{\left(r_3 \to r_3 - \frac{1}{2}r_2 \right)} \left[\begin{array}{ccc|c} 1 & -4 & 1 & -2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have $\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{A}|\mathbf{b}) = 2$. Therefore, the system $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solutions.

[7]

(ii) **Question** $$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & 4 \\ 1 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}.$$

Answer

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -2 & 3 & 4 & 5 \\ 1 & -1 & -3 & 0 \end{array}\right] \xrightarrow{\left(\begin{array}{c} r_2 \to r_2 + 2r_1 \\ r_3 \to r_3 - r_1 \end{array}\right)} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -1 & 2 & 11 \\ 0 & 1 & -2 & -3 \end{array}\right]$$

$$\xrightarrow{\left(r_3 \to r_3 + r_2 \right)} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -1 & 2 & 11 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

We have $\text{rank}\,(\mathbf{A}) = 2 \neq \text{rank}\,(\mathbf{A}|\mathbf{b}) = 3$. Therefore, the system $\mathbf{Ax} = \mathbf{b}$ has no solutions.

[7]

(b) **Question** *Find the general solution of whichever of the above systems is consistent and write the solution in parametric form*

Answer The general solution of the system (i) in parametric form is: $z = c$, $y = 3 + z = 3 + c$, $x = -2 + 4y - z = 10 + 3c$, where c is an arbitrary constant.

$$x = 10 + 3c,\ y = 3 + c,\ z = c.$$

[3]

Total for this question: 17 marks

2. (a) **Question** *Find the adjoint,* $\text{adj}(\mathbf{A})$*, determinant,* $\det \mathbf{A}$*, and inverse,* $\mathbf{A}^{-1}$*, of the square matrix*

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix}.$$

Answer The minors are:

$$\text{minors}(\mathbf{A}) = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -5 & -5 & -5 \\ -2 & -3 & -1 \\ -4 & -1 & -2 \end{bmatrix}$$

The cofactors are:

$$\text{cof}(\mathbf{A}) = \begin{bmatrix} -5 & 5 & -5 \\ 2 & -3 & 1 \\ -4 & 1 & -2 \end{bmatrix}$$

The adjoint matrix is:

$$\text{adj}(A) = \text{cof}(\mathbf{A})^T = \begin{bmatrix} -5 & 2 & -4 \\ 5 & -3 & 1 \\ -5 & 1 & -2 \end{bmatrix}$$

Check by multiplying:

$$\mathbf{A}\text{adj}(\mathbf{A}) = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 2 & -4 \\ 5 & -3 & 1 \\ -5 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = -5\mathbf{I},$$

thus

$$\det(\mathbf{A}) = -5$$

and

$$\mathbf{A}^{-1} = \frac{1}{2}\text{adj}(\mathbf{A}) = \begin{bmatrix} 1 & -0.4 & 0.8 \\ -1 & 0.6 & -0.2 \\ 1 & -0.2 & 0.4 \end{bmatrix}$$

[11]

(b) **Question** *Using your result from part (2a), find the solution to the system of simultaneous equations*

$$\begin{aligned} -x + 2z &= -1 \\ -x + 2y + 3z &= 4 \\ 2x + y - z &= 7 \end{aligned}$$

Answer

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}; \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -0.4 & 0.8 \\ -1 & 0.6 & -0.2 \\ 1 & -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

[6]

Total for this question: 17 marks

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3. *Consider the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & -2 \\ -3 & -1 & 3 \\ 1 & 1 & -3 \end{bmatrix}.$$

- **Question** *Write down its characteristic polynomial. Verify that its eigenvalues are* $\lambda_1 = 2$, $\lambda_2 = -2$ *and* $\lambda_3 = -4$.
  Answer Characteristic polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} -\lambda & -2 & -2 \\ -3 & -1-\lambda & 3 \\ 1 & 1 & -3-\lambda \end{bmatrix} = -\lambda^3 - 4\lambda^2 + 4\lambda + 16 = 0.$$

  Check that $\lambda = \{2, -2, -4\}$ are its roots:

$$\lambda = 2: \qquad -2^3 - 4*2^2 + 4*2 + 16 = -24 + 24 = 0. \qquad \checkmark$$

$$\lambda = -2: \qquad -(-2)^3 - 4*(-2)^2 + 4*(-2) + 16 = -24 + 24 = 0. \qquad \checkmark$$

$$\lambda = -4: \qquad -(-4)^3 - 4*(-4)^2 + 4*(-4) + 16 = -80 + 80 = 0. \qquad \checkmark$$

[5]

- **Question** *Find an eigenvector for each of the three eigenvalues.*
  Answer For $\underline{\lambda_1 = 2}$, equation $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} 0-2 & -2 & -2 \\ -3 & -1-2 & 3 \\ 1 & 1 & -3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -3 & -3 & 3 & 0 \\ 1 & 1 & -5 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  hence we may choose

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\underline{\lambda_2 = -2}$, equation $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} 2 & -2 & -2 \\ -3 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -3 & 1 & 3 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

hence we may choose

$$\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\underline{\lambda_3 = -4}$, equation $(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} 4 & -2 & -2 \\ -3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -3 & 3 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

hence we may choose

$$\mathbf{v}_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

[9]

- **Question** *Find the unit length (normalised) eigenvectors corresponding to the three eigenvalues λ_1, λ_2 and λ_3.*

  Answer The length of $\mathbf{v}_1$ is $\sqrt{1^2+1^2} = \sqrt{2}$, so $\mathbf{v}_1^{norm} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$, the length of $\mathbf{v}_2$ is $\sqrt{2}$, so $\mathbf{v}_2^{norm} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, the length of $\mathbf{v}_3$ is $\sqrt{2}$, so $\mathbf{v}_3^{norm} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

[3]

Total for this question: 17 marks

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4. (a) **Question** *Show that the point $(0,0)$ is a saddle point of the function*

$$f(x,y) = (1+x^2)\sin(xy).$$

Answer First derivatives test. The partial derivatives are

$$f_x = 2\sin(xy)x + (1+x^2)y\cos(xy); \qquad f_y = (1+x^2)x\cos(xy);$$

Both are zero for $x = 0$, $y = 0$, thus $(0,0)$ is a stationary point.
Second derivatives test. The second partial derivatives are

$$f_{xx} = 4y\cos(xy)x + 2\sin(xy) - (1+x^2)y^2\sin(xy);$$
$$f_{xy} = 2x^2\cos(xy) + (1+x^2)\cos(xy) - (1+x^2)y\sin(xy)x;$$
$$f_{yy} = -(1+x^2)x^2\sin(xy);$$

The determinant of their matrix at point $(0,0)$ is then

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Since $D < 0$, this is a saddle point.

[8]

(b) **Question** *Show that the point $(1,1)$ is a critical point of the function*

$$f(x,y) = x^2 - x + 1 - x\,y - y + y^2.$$

Classify this critical point.

Answer First derivatives test. The partial derivatives are

$$f_x = 2x - 1 - y; \qquad f_y = -x - 1 + 2y.$$

Both are zero for $x = 1$, $y = 1$, thus $(1,1)$ is a stationary point.
Second derivatives test. The second partial derivatives are

$$f_{xx} = 2; \qquad f_{xy} = -1; \qquad f_{yy} = 2.$$

The determinant of their matrix is then

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

Since $D > 0$, this is a local extremum. Since $f_{xx}(1,1) = 2 > 0$, this is a local minimum.

[9]

Total for this question: 17 marks

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5. (a) **Question** *Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} of the function*

$$f(x,y) = \ln(x)\sin(3y).$$

Answer

$$f_x = \frac{1}{x}\sin(3y), \qquad f_y = 3\ln(x)\cos(3y)$$

$$f_{xx} = -\frac{1}{x^2}\sin(3y), \qquad f_{xy} = \frac{3}{x}\cos(3y), \qquad f_{yy} = -9\ln(x)\sin(3y),$$

[4]

(b) **Question** *Using your result from part (5a), find the Taylor series at $(e, \pi/3)$ for f up to and including terms quadratic in the increments δx and δy.*

Answer Values of the function and its derivatives at the point $(e, \pi/3)$:

$$f = 0, \qquad f_x = 0, \qquad f_y = -3,$$

$$f_{xx} = 0, \qquad f_{xy} = -3/e, \qquad f_{yy} = 0,$$

Taylor's formula for the quadratic approximation

$$f(x + \delta x, y + \delta y) \approx f(x, y) + f_x(x, y)\delta x + f_y(x, y)\delta y$$

$$+ \frac{1}{2}\left(f_{xx}(x, y)(\delta x)^2 + 2f_{xy}\delta x\delta y + f_{yy}(x, y)(\delta y)^2\right)$$

Substituting here $x = e$, $y = \pi/3$ and the value of the function and its derivatives at this point, obtain

$$f(1 + \delta x, \pi/3 + \delta y) \approx -3\left(y - \frac{\pi}{3}\right) - \frac{3}{e}(x - e)\left(y - \frac{\pi}{3}\right)$$

[9]

(c) **Question** *Use the Taylor series found in part (5b) to obtain the linear and quadratic approximations for $f(2.5, 1.1)$, with 4 significant figures. For reference:*

$$\pi = 3.141592653\ldots, \qquad e = 2.718281828\ldots$$

Answer These values are obtained for $\delta x = 2.5 - e \approx -0.21828$ and $\delta y = 1.1 - \pi/3 \approx 0.052802$. Linear approximation:

$$f(2.5, 1.1) \approx -3\delta y \approx -0.1584(1)$$

Quadratic approximation:

$$f(2.5, 1.1) \approx -3\delta y - \frac{3}{e}\delta x\delta y \approx -0.1456(9)$$

(cf the exact value $f(2.5, 1.1) = -0.14454\ldots$ — optional self-control, no credit)

[4]

Total for this question: 17 marks

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6. (a) **Question** *The function $g(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion*

$$g(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right].$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

Answer

$$a_0 = \frac{1}{2}\int_{-1}^{1} g(x)\,\mathrm{d}x,$$

$$a_n = \int_{-1}^{1} g(x)\cos(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2\ldots,$$

$$b_n = \int_{-1}^{1} g(x)\sin(n\pi x)\,\mathrm{d}x\ n = 1, 2\ldots$$

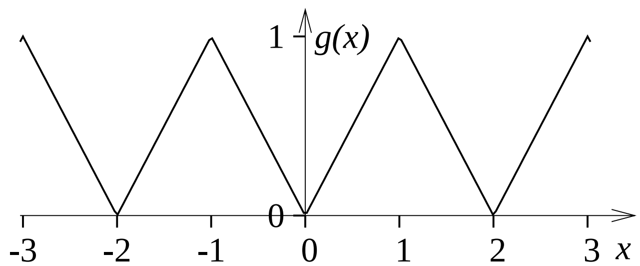
[5]

(b) **Question** *Sketch the graph of $g(x)$ defined by*

$$g(x) = \begin{cases} -x, & -1 \le x \le 0, \\ x, & 0 \le x \le 1, \\ g(x \pm 2), & \textit{for all } x, \end{cases}$$

for $-3 < x < 3$.

Answer



Question *Give the definition of an even function.*

Answer $f(-x) = f(x)$ for all x.

Question *Explain what special features a Fourier series of an even function has.*

Answer It lacks all sin terms.

Question *Explain why the function $g(x)$ defined above is even.*

Answer Graphical: the graph is mirror-symmetric about the vertical axis.
Analytical: it is even by definition within the symmetric interval $-1 \le x \le 1$, and periodic with period 2 equal to the length of that interval, therefore even everywhere.

[4]

(c) **Question** *Find the Fourier series of the function $g(x)$ defined above. You may use the following result:* $\int x\cos(kx)\,\mathrm{d}x = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx)$*, where $k \ne 0$ is a constant.*

Answer

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} g(x)\,\mathrm{d}x$$

$$= \frac{1}{2} \times 2 \int_0^1 x \, \mathrm{d}x$$

$$= \left(\frac{x^2}{2} \right)_0^1 = \frac{1}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} g(x) \cos\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x = \int_{-1}^{1} g(x) \cos(n\pi x) \, \mathrm{d}x$$

$$= 2 \int_0^1 g(x) \cos(n\pi x) \, \mathrm{d}x = 2 \int_0^1 x \cos(n\pi x) \, \mathrm{d}x$$

$$= 2 \left(\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right)_0^1 = 2 \, \frac{1}{n^2\pi^2} \, (\cos(n\pi) - 1)$$

$$= -\frac{2}{n^2\pi^2} \, (1 - (-1)^n) = \begin{cases} \frac{4}{n^2\pi^2}, & \text{for odd } n, \\ 0, & \text{for even } n, \end{cases}$$

Thus,

$$g(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5\ldots} \frac{1}{n^2} \cos(n\pi x)$$

Question *Write out this series explicitly up to terms with* $\cos(5\pi x)$ *and* $\sin(5\pi x)$*.*

Answer

$$g(x) \approx \frac{1}{2} - \frac{4}{\pi^2} \left(\cos(\pi x) + \frac{1}{9} \cos(3\pi x) + \frac{1}{25} \cos(5\pi x) \right)$$

[8]

Total for this question: 17 marks
