PAPER CODE NO.
MATH298

MAY 2005 EXAMINATIONS

Degree of Bachelor of Engineering : Year 2
Degree of Master of Engineering : Year 2

MATHEMATICS AND NUMERICAL METHODS FOR CIVIL ENGINEERS
(here section A only)

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to SIX questions,
of which no more than FIVE answers must be from the Section A and no less than ONE must be from the Section B.
Candidates should submit answers to Sections A and B in **separate books**.

SECTION A

1. (a) Use Gaussian elimination method to show that one of the following systems has no solutions and the other has an infinite number of solutions. (HINT: find the rank of **A** and **A**|**b** and compare with n, the number of unknowns):

(i) $$\begin{bmatrix} 1 & -4 & 1 \\ 1 & -2 & -1 \\ 2 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}.$$

(7 marks)

(ii) $$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & 4 \\ 1 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}.$$

(7 marks)

(b) Find the general solution of whichever of the above systems is consistent and write the solution in parametric form

(3 marks)

2. (a) Find the adjoint, adj $(\mathbf{A})$, determinant, $\det \mathbf{A}$, and inverse, $\mathbf{A}^{-1}$, of the square matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix}.$$

(11 marks)

(b) Using your result from part (2a), find the solution to the system of simultaneous equations

$$\begin{aligned} -x + 2z &= -1 \\ -x + 2y + 3z &= 4 \\ 2x + y - z &= 7 \end{aligned}$$

(6 marks)

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & -2 \\ -3 & -1 & 3 \\ 1 & 1 & -3 \end{bmatrix}.$$

- Write down its characteristic polynomial. Verify that its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -2$ and $\lambda_3 = -4$.

  (5 marks)

- Find an eigenvector for each of the three eigenvalues.

  (9 marks)

- Find the unit length (normalised) eigenvectors corresponding to the three eigenvalues λ_1, λ_2 and λ_3.

  (3 marks)

4. (a) Show that the point $(0, 0)$ is a saddle point of the function

$$f(x, y) = (1 + x^2)\sin(xy).$$

(8 marks)

(b) Show that the point $(1, 1)$ is a critical point of the function

$$f(x, y) = x^2 - x + 1 - x\,y - y + y^2.$$

Classify this critical point.

(9 marks)

5. (a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} of the function

$$f(x, y) = \ln(x)\sin(3y).$$

(4 marks)

(b) Using your result from part (5a), find the Taylor series at $(e, \pi/3)$ for f up to and including terms quadratic in the increments δx and δy.

(9 marks)

(c) Use the Taylor series found in part (5b) to obtain the linear and quadratic approximations for $f(2.5, 1.1)$, with 4 significant figures. For reference:

$$\pi = 3.141592653\ldots, \qquad e = 2.718281828\ldots$$

(4 marks)

6. (a) The function $g(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion

$$g(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right].$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

(5 marks)

(b) Sketch the graph of $g(x)$ defined by

$$g(x) = \begin{cases} -x, & -1 \le x \le 0, \\ x, & 0 \le x \le 1, \\ g(x \pm 2), & \text{for all } x, \end{cases}$$

for $-3 < x < 3$. Give the definition of an even function. Explain what special features a Fourier series of an even function has. Explain why the function $g(x)$ defined above is even.

(4 marks)

(c) Find the Fourier series of the function $g(x)$ defined above. You may use the following result: $\displaystyle\int x\cos(kx)\,\mathrm{d}x = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx)$, where $k \neq 0$ is a constant. Write out this series explicitly up to terms with $\cos(5\pi x)$ and $\sin(5\pi x)$.

(8 marks)

S E C T I O N B