

MATH298 May 2004 exam: solutions (part A)

All problems are similar to homework or class examples, except where stated explicitly as book-work.

1. (a) Use Gaussian elimination method to show that one of the following systems has no solutions and the other has an infinite number of solutions. (HINT: find the rank of \mathbf{A} and $\mathbf{A}|\mathbf{b}$ and compare with n , the number of unknowns):

$$(i) \begin{bmatrix} 1 & -4 & 1 \\ 1 & -2 & -3 \\ 2 & -7 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 9 \end{bmatrix}.$$

Answer

$$\begin{bmatrix} 1 & -4 & 1 & | & -2 \\ 1 & -2 & -3 & | & 4 \\ 2 & -7 & 0 & | & 9 \end{bmatrix} \xrightarrow{\begin{pmatrix} r_2 \rightarrow r_2 - r_1 \\ r_3 \rightarrow r_3 - 2r_1 \end{pmatrix}} \begin{bmatrix} 1 & -4 & 1 & | & -2 \\ 0 & 2 & -4 & | & 6 \\ 0 & 1 & -2 & | & 13 \end{bmatrix}$$

$$\xrightarrow{(r_3 \rightarrow r_3 - \frac{1}{2}r_2)} \begin{bmatrix} 1 & -4 & 1 & | & -2 \\ 0 & 2 & -4 & | & 6 \\ 0 & 0 & 0 & | & 10 \end{bmatrix}$$

We have $\text{rank}(\mathbf{A}) = 2 \neq \text{rank}(\mathbf{A}|\mathbf{b}) = 3$. Therefore, the system $\mathbf{Ax} = \mathbf{b}$ has no solutions.

7 marks for this part

$$(ii) \text{ Question } \begin{bmatrix} 1 & -3 & -1 \\ 2 & -5 & -4 \\ 1 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}.$$

Answer

$$\begin{bmatrix} 1 & -3 & -1 & | & 3 \\ 2 & -5 & -4 & | & 5 \\ 1 & -4 & 1 & | & 4 \end{bmatrix} \xrightarrow{\begin{pmatrix} r_2 \rightarrow r_2 - 2r_1 \\ r_3 \rightarrow r_3 - r_1 \end{pmatrix}} \begin{bmatrix} 1 & -3 & -1 & | & 3 \\ 0 & 1 & -2 & | & -1 \\ 0 & -1 & 2 & | & 1 \end{bmatrix}$$

$$\xrightarrow{(r_3 \rightarrow r_3 + r_2)} \begin{bmatrix} 1 & -3 & -1 & | & 3 \\ 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We have $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = 2$. Therefore, the system $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solutions.

7 marks for this part

- (b) **Question** Find the general solution of whichever of the above systems is consistent and write the solution in parametric form

Answer The general solution of the system (ii) in parametric form is: $z = c$, $y = -1 + 2z = -1 + 2c$, $x = 3 + z + 3y = 3 + c - 3 + 6c = 7c$, where c is an arbitrary constant.

$$x = 7c, y = -1 + 2c, z = c.$$

3 marks for this part

Total for this question: 17 marks

2. (a) **Question** Find the adjoint, $\text{adj}(\mathbf{A})$, determinant, $\det \mathbf{A}$, and inverse, \mathbf{A}^{-1} , of the square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Answer The minors are:

$$\text{minors}(\mathbf{A}) = \begin{bmatrix} \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -2 & 1 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

The cofactors are:

$$\text{cof}(\mathbf{A}) = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ -2 & -3 & -1 \end{bmatrix}$$

The adjoint matrix is:

$$\text{adj}(\mathbf{A}) = \text{cof}(\mathbf{A})^T = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix}$$

Check by multiplying:

$$\mathbf{A}\text{adj}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\mathbf{I},$$

thus

$$\det(\mathbf{A}) = 2$$

and

$$\mathbf{A}^{-1} = \frac{1}{2}\text{adj}(\mathbf{A}) = \begin{bmatrix} 0 & 1 & -1 \\ 1/2 & 1/2 & -3/2 \\ -1/2 & 1/2 & -1/2 \end{bmatrix}$$

11 marks for this part

- (b) **Question** Using your result from part (2a), find the solution to the system of simultaneous equations

$$\begin{aligned} x - 2z &= 1, \\ 2x - y - z &= -2, \\ x - y - z &= 3. \end{aligned}$$

Answer

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}; \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1/2 & 1/2 & -3/2 \\ -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ -3 \end{bmatrix}$$

6 marks for this part

Total for this question: 17 marks

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -3 & 0 & 4 \end{bmatrix}.$$

- **Question** Write down its characteristic polynomial. Verify that its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

Answer Characteristic polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 2 - \lambda & 1 \\ -3 & 0 & 4 - \lambda \end{vmatrix} = -\lambda(2 - \lambda)(4 - \lambda) + 3(2 - \lambda) = (2 - \lambda)(\lambda^2 - 4\lambda + 3) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

Check that $\lambda = \{1, 2, 3\}$ are its roots:

$$\lambda = 1: \quad -1^3 + 6 * 1^2 - 11 * 1 + 6 = -12 + 12 = 0. \quad \checkmark$$

$$\lambda = 2: \quad -2^3 + 6 * 2^2 - 11 * 2 + 6 = -30 + 30 = 0. \quad \checkmark$$

$$\lambda = 3: \quad -3^3 + 6 * 3^2 - 11 * 3 + 6 = -60 + 60 = 0. \quad \checkmark$$

5 marks for this part

- **Question** Find an eigenvector for each of the three eigenvalues.

Answer For $\lambda_1 = 1$, equation $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 2 - 1 & 1 \\ -3 & 0 & 4 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ -3 & 0 & 3 & 0 \end{array} \right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

hence we may choose

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\underline{\lambda_2 = 2}$, equation $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 2-2 & 1 \\ -3 & 0 & 4-2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
$$\Rightarrow \left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 \end{array} \right] \Leftrightarrow \left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

hence we may choose

$$\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\underline{\lambda_3 = 3}$, equation $(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{bmatrix} -3 & 0 & 1 \\ -1 & 2-3 & 1 \\ -3 & 0 & 4-3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
$$\Rightarrow \left[\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & 0 & 1 & 0 \end{array} \right] \Leftrightarrow \left[\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ 3 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Leftrightarrow \left[\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

hence we may choose

$$\mathbf{v}_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

9 marks for this part

- **Question** Find the unit length (normalised) eigenvectors corresponding to the three eigenvalues λ_1 , λ_2 and λ_3 .

Answer The length of \mathbf{v}_1 is $\sqrt{1^2 + 1^2} = \sqrt{2}$, so $\mathbf{v}_1^{norm} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$, the length of \mathbf{v}_2

is $\sqrt{1^2} = 1$, so $\mathbf{v}_2^{norm} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, the length of \mathbf{v}_3 is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so $\mathbf{v}_3^{norm} =$

$$\begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

3 marks for this part

Total for this question: 17 marks

4. (a) **Question** Show that the point $(0, 0)$ is a saddle point of the function

$$f(x, y) = (1 + x^2 - y)e^y.$$

Answer First derivatives test. The partial derivatives are

$$f_x = 2xe^y; \quad f_y = (x^2 - y)e^y;$$

Both are zero for $x = 0, y = 0$, thus $(0, 0)$ is a stationary point.

Second derivatives test. The second partial derivatives are

$$f_{xx} = 2e^y; \quad f_{xy} = 2xe^y; \quad f_{yy} = (-1 + x^2 - y)e^y;$$

The determinant of their matrix at point $(0, 0)$ is then

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2$$

Since $D < 0$, this is a saddle point.

8 marks for this part

- (b) **Question** Show that the point $(0, 0)$ is a critical point of the function

$$f(x, y) = x^2 + 2xy + 3y^2.$$

Classify this critical point.

Answer First derivatives test. The partial derivatives are

$$f_x = 2x + 2y; \quad f_y = 2x + 6y.$$

Both are zero for $x = 0, y = 0$, thus $(0, 0)$ is a stationary point.

Second derivatives test. The second partial derivatives are

$$f_{xx} = 2; \quad f_{xy} = 2; \quad f_{yy} = 6.$$

The determinant of their matrix is then

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix} = 12 - 4 = 8$$

Since $D > 0$, this is a local extremum. Since $f_{xx}(0, 0) = 2 > 0$, this is a local minimum.

9 marks for this part

Total for this question: 17 marks

5. (a) **Question** Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} of the function

$$f(x, y) = \cos(2x)e^{-y}.$$

Answer

$$\begin{aligned} f_x &= -2 \sin(2x)e^{-y}, & f_y &= -\cos(2x)e^{-y} \\ f_{xx} &= -4 \cos(2x)e^{-y}, & f_{xy} &= 2 \sin(2x)e^{-y}, & f_{yy} &= \cos(2x)e^{-y}, \end{aligned}$$

4 marks for this part

- (b) **Question** Using your result from part (5a), find the Taylor series at $(\pi/4, 0)$ for f up to and including terms quadratic in the increments δx and δy .

Answer Values of the function and its derivatives at the point $(\pi/4, 0)$:

$$f = 0, \quad f_x = -2, \quad f_y = 0$$
$$f_{xx} = 0, \quad f_{xy} = 2, \quad f_{yy} = 0,$$

Taylor's formula for the quadratic approximation

$$f(x + \delta x, y + \delta y) \approx f(x, y) + f_x(x, y)\delta x + f_y(x, y)\delta y$$
$$+ \frac{1}{2} (f_{xx}(x, y)(\delta x)^2 + 2f_{xy}\delta x\delta y + f_{yy}(x, y)(\delta y)^2)$$

Substituting here $x = \pi/4$, $y = 0$ and the value of the function and its derivatives at this point, obtain

$$f\left(\frac{\pi}{4} + \delta x, \delta y\right) \approx -2\delta x + 2\delta x \delta y$$

9 marks for this part

- (c) **Question** Use the Taylor series found in part (5b) to obtain the linear and quadratic approximations for $f(0.7, 0.1)$, with 4 decimal places. For reference: $\pi = 3.14159265358\dots$

Answer These values are obtained for $\delta x = 0.7 - \pi/4 \approx -0.08540$ and $\delta y = 0.1$. Linear approximation:

$$f(0.7, 0.1) \approx -2\delta x \approx 0.1708$$

Quadratic approximation:

$$f(0.7, 0.1) \approx -2\delta x + 2\delta x \delta y \approx 2 * (-0.08540) * (-1 + 0.1) \approx 0.1537$$

(cf the exact value $f(0.7, 0.1) = 0.15379\dots$ — optional self-control, no credit)

4 marks for this part

Total for this question: 17 marks

6. (a) **Question** The function $g(x)$ is periodic, with period $p = 2L = 4$, and has the Fourier series expansion

$$g(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right].$$

State the formulae for the Fourier coefficients, a_0 , a_n , $n = 1, 2, \dots$ and b_n , $n = 1, 2, \dots$, valid for this period.

Answer

$$a_0 = \frac{1}{4} \int_{-2}^2 g(x) dx,$$

$$a_n = \frac{1}{2} \int_{-2}^2 g(x) \cos\left(\frac{n\pi x}{2}\right) dx, \quad n = 1, 2, \dots,$$

$$b_n = \frac{1}{2} \int_{-2}^2 g(x) \sin\left(\frac{n\pi x}{2}\right) dx, \quad n = 1, 2, \dots$$

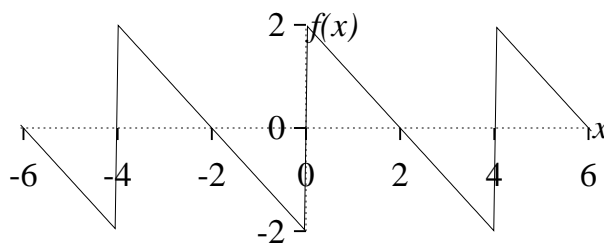
5 marks for this part

(b) **Question** Sketch the graph of $g(x)$ defined by

$$g(x) = \begin{cases} -2 - x, & -2 < x < 0, \\ 2 - x, & 0 < x < 2, \\ g(x \pm 4), & \text{for all } x, \end{cases}$$

for $-6 < x < 6$.

Answer



Question Give the definition of an odd function.

Answer $f(-x) = -f(x)$ for all x .

Question Explain what special features a Fourier series of an odd function has.

Answer $a_0 = a_n = 0$ for all n .

Question Explain why the function $g(x)$ defined above is odd.

Answer Graphical: the graph is center-symmetric about the origin.

Analytical: it is odd by definition within the symmetric interval $-2 \leq x \leq 2$, and periodic with period 4 equal to the length of that interval, therefore odd everywhere.

4 marks for this part

(c) **Question** Find the Fourier series of the function $g(x)$ defined above. You may use the following result: $\int (A + Bx) \sin(kx) dx = -\frac{A + Bx}{k} \cos(kx) + \frac{B}{k^2} \sin(kx)$, where A , B and $k \neq 0$ are constants.

Answer

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_{-2}^2 g(x) \sin(n\pi x/2) dx \\ &= \int_0^2 g(x) \sin(n\pi x/2) dx \end{aligned}$$

$$= \int_0^2 (2-x) \sin(n\pi x/2) dx$$

(now use the provided ready answer for the indefinite integral, for $A = 2$, $B = -1$, $k = n\pi/2$)

$$\begin{aligned} &= \left(-\frac{2-x}{(n\pi/2)} \cos(n\pi x/2) - \frac{1}{(n\pi/2)^2} \sin(n\pi x/2) \right)_0^2 \\ &= -\frac{2-2}{n\pi/2} \cos(n\pi 2/2) + \frac{2-0}{n\pi/2} \cos(n\pi 0/2) = \frac{4}{n\pi} \end{aligned}$$

Thus,

$$g(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x/2)$$

Question Write out this series explicitly up to terms with $\cos(5\pi x)$ and $\sin(5\pi x)$.

Answer

$$g(x) \approx \frac{4}{\pi} \left(\sin\left(\frac{\pi x}{2}\right) + \frac{1}{2} \sin(\pi x) + \frac{1}{3} \sin\left(\frac{3\pi x}{2}\right) + \frac{1}{4} \sin(2\pi x) + \frac{1}{5} \sin\left(\frac{5\pi x}{2}\right) \right)$$

8 marks for this part

Total for this question: 17 marks
