MAY 2004 EXAMINATIONS

Degree of Bachelor of Engineering : Year 2
Degree of Master of Engineering : Year 2

MATHEMATICS AND NUMERICAL METHODS FOR CIVIL ENGINEERS
(here section A only)

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to SIX questions,
of which no more than FIVE answers must be from the Section A and no less than ONE must be from the Section B.
Candidates should submit answers to Sections A and B in **separate books**.

S E C T I O N   A

1. (a) Use Gaussian elimination method to show that one of the following systems has no solutions and the other has an infinite number of solutions. (HINT: find the rank of **A** and **A**|**b** and compare with n, the number of unknowns):

(i) $$\begin{bmatrix} 1 & -4 & 1 \\ 1 & -2 & -3 \\ 2 & -7 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 9 \end{bmatrix}.$$

[7 marks]

(ii) $$\begin{bmatrix} 1 & -3 & -1 \\ 2 & -5 & -4 \\ 1 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}.$$

[7 marks]

(b) Find the general solution of whichever of the above systems is consistent and write the solution in parametric form

[3 marks]

2. (a) Find the adjoint, adj ($\mathbf{A}$), determinant, det $\mathbf{A}$, and inverse, $\mathbf{A}^{-1}$, of the square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

[11 marks]

(b) Using your result from part (2a), find the solution to the system of simultaneous equations

$$\begin{aligned} x - 2z &= 1, \\ 2x - y - z &= -2, \\ x - y - z &= 3. \end{aligned}$$

[6 marks]

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -3 & 0 & 4 \end{bmatrix}.$$

- Write down its characteristic polynomial. Verify that its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

[5 marks]

- Find an eigenvector for each of the three eigenvalues.

[9 marks]

- Find the unit length (normalised) eigenvectors corresponding to the three eigenvalues λ_1, λ_2 and λ_3.

[3 marks]

4. (a) Show that the point $(0, 0)$ is a saddle point of the function

$$f(x, y) = (1 + x^2 - y)e^y.$$

[8 marks]

(b) Show that the point $(0, 0)$ is a critical point of the function

$$f(x, y) = x^2 + 2xy + 3y^2.$$

Classify this critical point.

[9 marks]

5. (a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} of the function

$$f(x,y) = \cos(2x)e^{-y}.$$

[4 marks]

(b) Using your result from part (5a), find the Taylor series at $(\pi/4, 0)$ for f up to and including terms quadratic in the increments δx and δy.

[9 marks]

(c) Use the Taylor series found in part (5b) to obtain the linear and quadratic approximations for $f(0.7, 0.1)$, with 4 decimal places. For reference: $\pi = 3.14159265358\ldots$.

[4 marks]

6. (a) The function $g(x)$ is periodic, with period $p = 2L = 4$, and has the Fourier series expansion

$$g(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right].$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

[5 marks]

(b) Sketch the graph of $g(x)$ defined by

$$g(x) = \begin{cases} -2 - x, & -2 < x < 0, \\ 2 - x, & 0 < x < 2, \\ g(x \pm 4), & \text{for all } x, \end{cases}$$

for $-6 < x < 6$. Give the definition of an odd function. Explain what special features a Fourier series of an odd function has. Explain why the function $g(x)$ defined above is odd.

[4 marks]

(c) Find the Fourier series of the function $g(x)$ defined above. You may use the following result: $\int (A + Bx)\sin(kx)\,\mathrm{d}x = -\frac{A + Bx}{k}\cos(kx) + \frac{B}{k^2}\sin(kx)$, where A, B and $k \neq 0$ are constants. Write out this series explicitly up to terms with $\cos(5\pi x)$ and $\sin(5\pi x)$.

[8 marks]

SECTION B