Math 295

January 2005 Exam

Engineering Analysis — Year 2 Engineers

Full marks will be awarded for complete answers to FOUR questions. Only the best 4 answers will be taken into account. Note that each question carries a total of 20 marks that are distributed as stated.

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For this half unit course, there is a C/A component.
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1. (**20 marks**)

(a) A common practice for making engineering products is by punching desirable shapes out of sheet metal. Suppose one wishes to punch out, on a square metal sheet of size s (m) and thickness t (m), 4 identical holes of circular shapes of radius r (m).

(i) Show that the remaining sheet has the volume of

$$V = V(r, s, t) = ts^2 - 4\pi tr^2 \ (m^3).$$

[5 marks]

(ii) If all measurements are subject to the operational error of $\delta = 0.01$ (m), determine the approximate error (keeping at least 3 decimal digits) in the above quantity V, in the case of these specifications:

$$t = 1 \ (m), \qquad s = 3 \ (m), \qquad r = 0.5 \ (m).$$

[9 marks]

(b) For the following function

$$f = f(x, y, z) = x\sin(y + 3z) + \cos(y) + ye^{-x},$$

(i) verify that [3 marks]

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x};$$

(ii) show that f satisfies the following equation [3 marks]

$$\frac{\partial}{\partial y}\left(e^x \frac{\partial f}{\partial x}\right) = e^x \cos(y + 3z) - 1.$$

2. (**20 marks**)

Consider the following implicit function $z = z(x, y)$ determined by

$$\tan^{-1}\left(\sin^2(3xy)\right) + z^2 - 6xyz - 12y - 10 = 0.$$

Given the formula

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2},$$

(a) use the implicit differentiation to compute z_x and verify that

$$z_y = 3\frac{x\sin(3\,xy)\cos(3\,xy) - (xz+2)\left[1+\sin^4(3\,xy)\right]}{\left[1+\sin^4(3\,xy)\right](3\,xy - z)};$$

[10 marks]

(b) evaluate z_x and z_y at the point $P_0 = (x_0, y_0, z_0) = (0, -1/2, 2)$ and hence find the tangent plane to this surface at P_0.

[10 marks]

3. (**20 marks**)

(a) For the function $f = f(x, y) = (y-1)^4 e^{-x^2}$, near $p_1 = (x_1, y_1) = (0, 2)$, find the Taylor formulae $p_1(x, y)$ and $p_2(x, y)$ which are respectively (up to and including) the first and second order terms.

[14 marks]

(b) For the function $F = z^3 f(x, y) = z^3 (y-1)^4 e^{-x^2}$, compute the direction derivative $D_{t_u} F$ at the point $p_2 = (x_2, y_2, z_2) = (0, 2, 1)$ in the direction

$$t = (3, 4, 5).$$

[6 marks]

4. (**20 marks**)

Given the periodic function $f(x)$, with period $L = 2\pi$,

$$f(x) = \begin{cases} \pi + x, & -\pi \le x < 0, \\ \pi - x, & 0 \le x \le \pi, \end{cases}$$

(a) sketch the function $f(x)$ in $-3\pi < x \le 3\pi$ and decide if $f(x)$ is odd or even.

[4 marks]

(b) Find the Fourier series for the above function.

Hint. You may assume $\cos(n\pi) = (-1)^n$. [12 marks]

(c) To obtain a Fourier half range series, extend the following function

$$f(x) = \begin{cases} (\pi/2 - x)x, & 0 \le x < \pi/2, \\ x - \pi/2, & \pi/2 \le x \le \pi, \end{cases}$$

as shown in Figure 1 to an even function $g(x)$ in $[-\pi, \pi]$ and sketch $g(x)$.

(*Only the sketch is required. Do not work out the Fourier series.*) [4 marks]

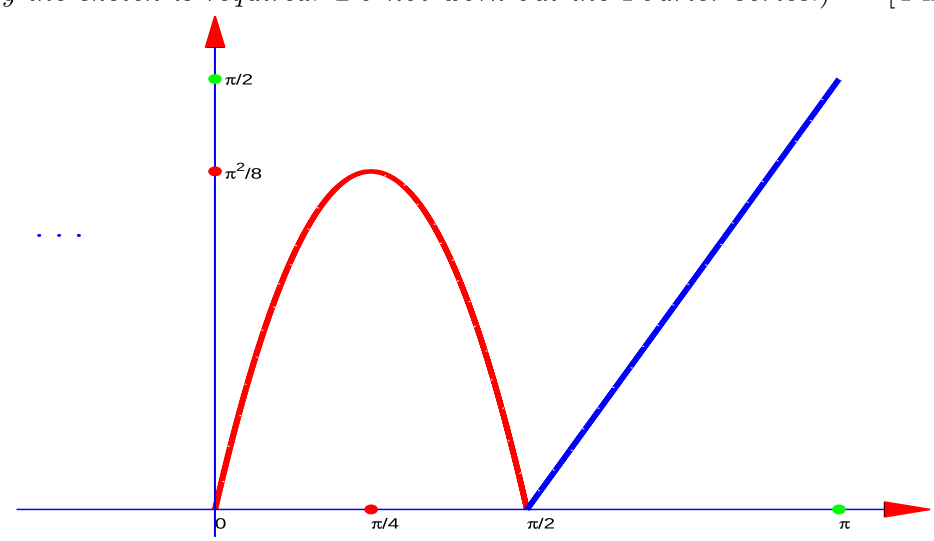

Figure 1. The function $f(x)$ in $[0, \pi]$ — Q4 (c)

5. (**20 marks**)

Define the Laplace transform for function $f(t)$. [2 marks]

(a) Using the definition, compute

$$\mathcal{L}\{e^{\mu_1 t} + e^{-\mu_2 t}\}$$

where μ_1, μ_2 are some real and known constants. [4 marks]

(b) Consider the linear ordinary differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t}, \qquad x(0) = -2, \quad \frac{dx}{dt}(0) = 3.$$

Firstly verify that

$$\overline{x}(s) = -\frac{(3+2s)s}{(s+1)^3}.$$

Secondly find $x = x(t)$ by using the inverse Laplace transform approach and the properties of the Laplace transform:

$$\begin{cases} \mathcal{L}\{\frac{dx}{dt}(t)\} = s\mathcal{L}\{x(t)\} - x(0), \\ \mathcal{L}\{\frac{d^2x}{dt^2}(t)\} = s^2\mathcal{L}\{x(t)\} - sx(0) - \frac{dx}{dt}(0), \\ \mathcal{L}\{t^n e^{\alpha t}\} = \frac{n!}{(s-\alpha)^{n+1}}. \end{cases}$$

[11 marks]

(c) Without solving any equations, decide if the Laplace transforms may be applied to solve the following equations (simply answer Yes or No):

$$(3a) \qquad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = \sin^2(t)e^{-t},$$

$$(3b) \qquad \left(\frac{d^2x}{dt^2}\right)^2 + 2\frac{dx}{dt} + x = e^{-t},$$

$$(3c) \qquad \frac{d^2x}{dt^2} + 2x\frac{dx}{dt} + x = e^{-t}.$$

[3 marks]