PAPER CODE NO.
MATH295

THE UNIVERSITY
of LIVERPOOL

January 2007 EXAMINATIONS

Bachelor of Engineering: Year 2
Master of Engineering: Year 2

ENGINEERING ANALYSIS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions. Only the best FOUR questions will be counted.

This examination contributes 90% towards the final mark.
The balance comes from two class tests.

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1.

(a) Given

$$u(x,y) = \ln(x+y) + \tan^{-1}(xy),$$

and

$$x = \tan(t), \quad y = \sqrt{t},$$

express du/dt as a function of x, y and t by using the chain rule.

[7 marks]

(b) Given

$$u^2 - xu - yu^3 + 1 = 0,$$

evaluate u_{xx} by the use of implicit differentiation.

[9 marks]

(c) Given $f(x,y) = \mathrm{e}^{\sin(x-y)}$, obtain its Taylor series expansion about the point $(x,y) = (1,1)$, up to and including second order terms.

[9 marks]

Hint: Use the formula

$$f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{1}{2}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

for the expansion of the function $f(x,y)$ about the point (a,b) up to and including second order terms.

2.

(a) Find the two stationary points of the function

$$f(x,y) = x^3 - 6xy + y^3,$$

and classify them.

[12 marks]

Hint: A stationary point (a,b) of the function f(x,y) is
a minimum if $f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b) < 0$ and $f_{xx}(a,b) > 0$,
a maximum if $f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b) < 0$ and $f_{xx}(a,b) < 0$,
and a saddle point if $f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b) > 0$.

(b) Find the directional derivative of the function

$$f(x,y,z) = z\cos(xy)$$

at the point $P(0,1,5)$ along the direction from P to $P'(6,3,8)$.
Also find the equation of the tangent plane to the surface

$$F(x,y,z(x,y)) = f(x,y,z) - 5 = 0$$

together with the parametric equations of its normal at the point $P(0,1,5)$.

[13 marks]

Hint: The directional derivative of the function $f = f(x,y,z)$ at the point $P(a,b,c)$ in the direction of the unit vector $\mathbf{t}_u$ is

$$D_{\mathbf{t}_u} f = \operatorname{grad} f\Big|_{P(a,b,c)} \cdot \mathbf{t}_u.$$

The equation of the tangent plane to the surface $F(x,y,z(x,y)) = 0$ at the point $P(a,b,c)$ is given by

$$(x-a, y-b, z-c) \cdot \operatorname{grad} F\Big|_{P(a,b,c)} = 0$$

and the parametric equations of the normal at the same point are

$$(x,y,z) = (a,b,c) + \lambda \operatorname{grad} F\Big|_{P(a,b,c)}.$$

3. The formula for the Fourier series of a function $f(t)$, $t \in (0, 2\pi)$, of period 2π is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nt) + b_n \sin(nt)\},$$

where the Euler-Fourier coefficients can be computed by the formulae

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t) dt, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt) dt.$$

(a) Given that

$$f(t) = \begin{cases} \pi - t, & 0 < t < \pi, \\ 0, & \pi \leq t < 2\pi, \end{cases}$$

sketch the sum of its Fourier series for $-4\pi < t < 4\pi$.

[5 marks]

(b) Show that its Fourier series has the following form

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1 - (-1)^n}{\pi n^2} \cos(nt) + \frac{1}{n} \sin(nt) \right\}.$$

[10 marks]

(c) Hence, by setting $t = 0$, prove that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

[5 marks]

(d) Given that

$$F(t) = \begin{cases} \frac{\pi}{2} - t, & 0 < t < \pi, \\ -\frac{\pi}{2}, & \pi \leq t < 2\pi, \end{cases}$$

obtain its Fourier series directly from (a) and (b).

[5 marks]

4. The formula for the Fourier half range cosine series of a function $f(t)$, $t \in (0,\alpha)$, of period 2α is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\alpha}t\right),$$

where

$$a_0 = \frac{2}{\alpha}\int_0^{\alpha} f(t)dt, \quad a_n = \frac{2}{\alpha}\int_0^{\alpha} f(t)\cos\left(\frac{n\pi}{\alpha}t\right)dt.$$

The formula for the Fourier half range sine series of a function $f(t)$, $t \in (0,\alpha)$, of period 2α is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\alpha}t\right), \quad \text{where} \quad b_n = \frac{2}{\alpha}\int_0^{\alpha} f(t)\sin\left(\frac{n\pi}{\alpha}t\right)dt.$$

(a) Explain briefly, with reference to the coefficients a_0, a_n or b_n, why the constant 2 appears in their formulae.

[3 marks]

(b) Given that

$$f(t) = \begin{cases} 1, & 0 < t < L/2, \\ -1, & L/2 < t < L, \end{cases}$$

sketch the sum of its Fourier half range sine and the sum of its Fourier half range cosine series for $-3L < t < 3L$.

[10 marks]

(c) Show that its Fourier half range sine series has the following form

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 + (-1)^n - 2\cos(\frac{n\pi}{2})}{n}\sin(\frac{n\pi}{L}t).$$

[5 marks]

(d) Show that its Fourier half range cosine series has the following form

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n}\cos(\frac{n\pi}{L}t).$$

[7 marks]

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5. Given that the functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$2\frac{dx}{dt} + 2x + \frac{dy}{dt} = 2t + 4, \quad 4x - \frac{dx}{dt} + 2y = 8t + 7$$

and the initial conditions $x(0) = 2$, $y(0) = -1$, show that the Laplace transform of $x(t)$ is

$$\bar{x}(s) = \frac{2s^3 - s^2 + 4}{s^2(s^2 + 4)}.$$

[10 marks]

Find the unknown coefficients in the equation

$$\frac{2s^3 - s^2 + 4}{s^2(s^2 + 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4}$$

[7 marks]

and hence, or otherwise, find the functions $x(t)$ and $y(t)$.

[8 marks]

Hint: Use the table of Laplace transforms on the separate sheet.