1. (a) The longitudinal wave speed in a rod with Lamé constants λ and μ with mass density ρ is given by the formula

$$c_L(\lambda, \mu, \rho) = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

If the measurements of λ, μ and ρ are subject to changes of $d\lambda$, $d\mu$ and $d\rho$ respectively, show that the proportional change in c_L is given by

$$\frac{dc_L}{c_L} = \frac{1}{2}\left[\frac{1}{\lambda + 2\mu}d\lambda + \frac{2}{\lambda + 2\mu}d\mu - \frac{d\rho}{\rho}\right].$$

[5 marks]

Find the percentage error in the measurement of c_L if the maximum percentage errors in λ, μ and ρ are 2%, 3% and 1% respectively and the values obtained experimentally are $\lambda = 1$, $\mu = 1/2$ and $\rho = 2$. [5 marks]

(b) Find ALL the stationary points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2,$$

and for each determine whether it is a maximum, a minimum or a saddle point. [10 marks]

2. Consider the implicit function $z = z(x, y)$ determined by

$$F(x, y, z) = z^2 + zx^2y + 6xy - \log(x^2 + y^2) = 0.$$

(a) Use implicit differentiation to compute z_x and verify that

$$z_y = \frac{2y - (x^2 + y^2)(x^2z + 6x)}{(x^2 + y^2)(2z + yx^2)}.$$

[6 marks]

(b) Evaluate z_x and z_y at the point $P_0 = (0, e, \sqrt{2})$ and hence find the equation of the tanget plane to the surface $z = z(x, y)$ at P_0. [4 marks]

(c) When is z NOT defined as an implicit function of x and y? [4 marks]

(d) Considering the change of variable $x = \cos(t)$, $y = \sin(t)$ and $z = t$, find $\frac{dF}{dt}$ using the Chain Rule. [6 marks]

3. (a) For the function $f(x,y) = y\cos(x^2y)$ find the Taylor polynomial of second order $p_2(x,y)$ near $(1, 2\pi)$, verifying first that

$$f_{xy}(x,y) = -4xy\sin(x^2y) - 2x^3y^2\cos(x^2y)$$
$$f_{xy}(1,2\pi) = -8\pi^2, \quad f_y(1,2\pi) = 1.$$

[14 marks]

(b) Given the function

$$F(x,y,z) = x^2\cos(y+2z) + y^2ze^{-x},$$

compute the directional derivative $D_{t_u}F$ at the point $P_0 = (2, 0, \pi)$ in the direction $\mathbf{t} = (1, -1, 3)$. [6 marks]

4. (a) Sketch the graph of the function $f(t)$ in the interval $-4\pi < t \leq 4\pi$ when

$$f(t) = |t|, \quad -\pi < t \leq \pi,$$

is periodic with period $T = 2\pi$. [4 marks]

(b) State the formulae for all the Fourier coefficients a_0, a_n and b_n for the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt)).$$

[6 marks]

(c) Verify that the Fourier series for the above function is

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi}\cos((2n-1)t).$$

You may assume that $\cos(n\pi) = (-1)^n$. [10 marks]

5. Using the definition of the Laplace transform

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,$$

(a) Show that if $|x(t)| \leq Me^{bt}$ for $t \geq t_0$, $x(t)$ continuous for $t \geq 0$ and $\frac{dx}{dt}(t)$ piecewise continuous, then for $s > b$

$$\mathcal{L}\{\frac{dx}{dt}(t)\} = s\mathcal{L}\{x(t)\} - x(0).$$

[8 marks]

(b) Solve the ordinary differential equation for $x = x(t)$

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = e^{-2t}, \quad x(0) = 1, \frac{dx}{dt}(0) = -1,$$

by using the Laplace transform, verifying first that

$$\overline{x}(s) = \frac{s^2 - 2s - 7}{(s+2)(s^2 - 3s - 4)},$$

and using the fact in (a) together with the following properties of the Laplace transform:

$$\begin{cases} \mathcal{L}\{\frac{d^2x}{dt}(t^2)\} = s^2\mathcal{L}\{x(t)\} - sx(0) - \frac{dx}{dt}(0), \\ \mathcal{L}\{t^n e^{\alpha t}\} = \frac{n!}{(s-\alpha)^{n+1}}. \end{cases}$$

[12 marks]