PAPER CODE NO.
MATH295

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2003 EXAMINATIONS

Bachelor of Engineering: Year 2
Master of Engineering: Year 2

ENGINEERING ANALYSIS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions. Only the best FOUR questions will be counted.

This examination contributes 80% towards the final mark.
The balance comes from two class tests.

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1.

(a) Given

$$f(x, y) = xy^2 \sin(x/y)$$

evaluate the first order derivatives f_x and f_y.

[7 marks]

(b) Given

$$u(x, y) = x^2 + \mathrm{e}^x \sin(y)$$

and

$$x = \tan(t), \quad y = t^3,$$

express $\dfrac{du}{dt}$ as a function of x, y and t by using the chain rule and hence evaluate $\dfrac{du}{dt}$ at $t = \pi$.

[10 marks]

(c) Given

$$xu^2 - yu^2 + xyu - 3 = 0$$

evaluate $\dfrac{\partial u}{\partial y}$ by the use of implicit differentiation.

[8 marks]

2.

(a) Find the critical points of the function

$$f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$$

and classify them.

[10 marks]

Hint: A critical point of the function f(x,y) is a minimum if

$$\left(f_{xy}\Big|_P\right)^2 - f_{xx}\Big|_P f_{yy}\Big|_P < 0 \quad \textit{and} \quad f_{xx}\Big|_P > 0,$$

a maximum if

$$\left(f_{xy}\Big|_P\right)^2 - f_{xx}\Big|_P f_{yy}\Big|_P < 0 \quad \textit{and} \quad f_{xx}\Big|_P < 0,$$

and a saddle point if

$$\left(f_{xy}\Big|_P\right)^2 - f_{xx}\Big|_P f_{yy}\Big|_P > 0.$$

(b) Given

$$f(x,y) = \ln(x^2 + y^2),$$

evaluate f_x, f_y, f_{xx}, f_{xy} and f_{yy} at the point (x,y) = (1,0).

Hence, show that the Taylor series expansion of the function about the point $(x,y) = (1,0)$, up to and including second order terms, is

$$2x - 2 - (x-1)^2 + y^2.$$

[15 marks]

Hint: Use the formula

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{1}{2}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

for the expansion of the function $f(x,y)$ about the point $(x,y) = (a,b)$.

3. The formula for the Fourier series of a function $f(t)$, $t \in (0, 2\pi)$, of period 2π is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos(nt) + b_n \sin(nt)\},$$

where the Euler-Fourier coefficients can be computed by the formulae

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)dt, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)dt.$$

(a) Given that

$$f(t) = \begin{cases} -k, & 0 < t < \pi/2, \\ 0, & \pi/2 \le t < 2\pi, \end{cases} \qquad k \text{ is a constant},$$

sketch the sum of its Fourier series for $-4\pi < t < 4\pi$.

[5 marks]

(b) Show that its Fourier series has the following form

$$f(t) = -\frac{k}{4} - \frac{k}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\pi/2)}{n}\cos(nt) - \frac{\cos(n\pi/2) - 1}{n}\sin(nt)\right\}.$$

[10 marks]

(c) Hence, by choosing $k = 1$ and setting $t = \pi/2$, prove that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)} = \frac{\pi}{4}.$$

[5 marks]

(d) Given that

$$F(t) = \begin{cases} -k/2, & 0 < t < \pi/2, \\ k/2, & \pi/2 \le t < 2\pi, \end{cases} \qquad k \text{ is a constant},$$

obtain its Fourier series directly from (a) and (b)? Is $F(t)$ an odd or an even function?

[5 marks]

4. The formula for the Fourier series of a function $f(t)$, $t \in (-\alpha, \alpha)$, of period 2α is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{\alpha}t\right) + b_n \sin\left(\frac{n\pi}{\alpha}t\right) \right\},$$

where the Euler-Fourier coefficients can be computed by the formulae

$$a_0 = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} f(t)dt, \quad a_n = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} f(t)\cos\left(\frac{n\pi}{\alpha}t\right) dt, \quad b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\sin\left(\frac{n\pi}{\alpha}t\right) dt.$$

(a) Given that

$$f(t) = \begin{cases} t, & 0 < t < L/2, \\ L - t, & L/2 \le t < L, \end{cases}$$

sketch the sum of its Fourier half range Sine and the sum of its Fourier half range Cosine series for $-3L < t < 3L$.

[5 marks]

(b) Show that its Fourier half range Sine series has the following form

$$f(t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \sin\left(\frac{n\pi}{L}t\right).$$